1. Oscillation
    1. Periodic motion
    2. Simple harmonic motion
    3. SHM equation
    4. Linear SHM
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    6. Simple and physical pendulum
    7. Composition of harmonic motions
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2. Waves
    1. Waves
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Periodic motion

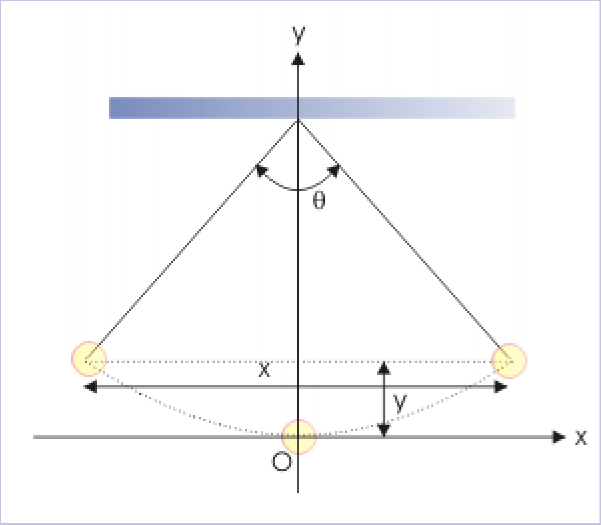
Periodic phenomena or event repeats after certain interval like seasons, motion of planets, swings, pendulum etc. In this module, our focus, however, is the physical movement of particle or body, which shows a pattern recurring after certain fixed time interval.

Representation of periodic motion has a basic pattern, which is repeated at regular intervals. What it means that if we know the basic form (building block), then we can describe the motion by following the pattern again and again.

Periodic attributes

A periodic motion can be described with respect to different quantities. A given periodic motion can have a host of attributes which may undergo periodic variations. Consider, for example, the case of a pendulum. We can choose any of the attributes like angle (θ), horizontal displacement (x), vertical displacement (y), kinetic energy (K), potential energy (U) etc. The values of these quantities undergo periodic alteration with respect to time. These attributes constitute periodic attributes of the periodic motion.

Attributes of periodic motion



Attributes undergo periodic alteration with respect to time.

There are, however, other attributes, which may remain constant during periodic motion. If we consider pendulum executing simple harmonic motion (it is a particular periodic motion), then total mechanical energy of the system is constant and as such is independent of time. Hence, we need to pick appropriate attribute(s) to describe a periodic motion in accordance with problem situation in hand.

Description of periodic motion

We need a mathematical model to describe periodic motion. For this, we employ certain mathematical functions. The important feature of a periodic function is that its value is repeated after certain interval. We call this interval as “period”. In case, this period refers to time, then the same is called “time period”. Mathematically, a periodic function is defined as :

Periodic motion

> A function is said to be periodic if there exists a positive real number “T”, which is independent of “t”, such that $f(t+T)=f(t)$.

The least positive real number “T” (T>0) is known as the fundamental period or simply the period of the function. The “T” is not a unique positive number. All integral multiple of “T” is also the period of the function.

In the context of periodic function, an “aperiodic” function is one, which in not periodic. On the other hand, a function is said to be anti-periodic if :

$$f(t+T)=-f(t)$$

Example

Problem 1: Prove that "$\sin t$" is a periodic function. Also find its period.

Solution :

We know that :

$$\sin(n\pi + t) = (-1)^n \sin t, \quad \text{when “n” is an integer.}$$

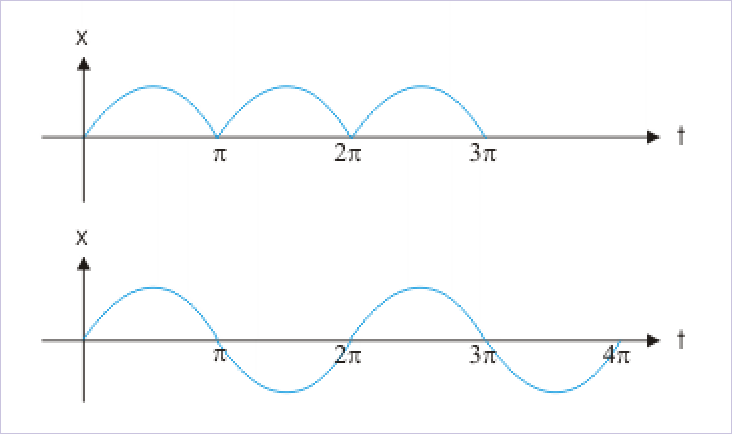
$$\sin(n\pi + t) = \sin t, \quad \text{when “n” is an even integer.}$$

Thus, there exists T>0 such that f(t+T) = f(t). Further “nπ” is independent of “t”. Hence, “sin t” is a periodic function. Its period is the least value, when n = 2 (first even positive integer),

$$\Rightarrow T = 2\pi$$

The period of "sine" function is collaborated from the figure also. Note that we can not build a sine curve with a half cycle of period “π” (upper figure). We require a full cycle of “2π” to build a sine curve (lower figure).

Period of sine function



A full cycle of “2π” is used to build a sine curve (lower figure).

Elements of periodic motion

Here, we describe certain important attributes of periodic motion, which are extensively used to describe a periodic motion. Though, it is expected that readers are already familiar with these terms, but we present the same for the sake of completeness.

1: Time period (T) : It is the time after which a periodic motion (i.e. the pattern of motion) repeats itself. Its dimensional formula is [T] and unit is “second” in SI unit.

2: Frequency (n, ν) : It is the number of times the unit of periodic motion is repeated in unit time. In SI system, its unit is s^{-1}, which is known as "Hertz" or "Hz" in short. An equivalent name is cycles per second (cps). Its dimensional formula is $[T^{-1}]$. Time period and frequency are inverse to each other :

$$T = \frac{1}{\nu}$$

3: Angular frequency (ω) : It is the product of “2π” and frequency “ν”. In the case of rotational motion, angular frequency is equal to the angle (radian) described per unit time (second) and is equal to the magnitude of average angular velocity.

$$\omega = 2\pi\nu$$

The unit of angular frequency is radian/s. In general, angular frequency and angular velocity are referred in equivalent manner. However, we should emphasize that we refer only the magnitude of angular velocity, when quoted to mean frequency. Also,

$$\omega = \frac{2\pi}{T}$$

Since “2π” is a constant, the dimensional formula of angular frequency is same as that of frequency i.e. $[T^{-1}]$.

4: Displacement (x or y) : It is equal to change in the physical quantity in periodic motion. This physical quantity can be any thing like displacement, electric current, pressure etc. The unit of displacement obviously depends on the physical quantity under consideration.

Period of periodic motion

There are many different periodic functions. In our course, however, we shall be dealing mostly with trigonometric functions. Some important results about period are useful in finding period of a given function.

1: All trigonometric functions are periodic.

2: The periods of sine, cosine, secant and cosecant functions are “2π”, whereas periods of tangent and cotangent functions are “π”.

3: If "k","a" and "b" are positive real values and “T” be the period of periodic function “f(x)”, then :

- "kf(x)" is periodic with period “T”.
- "f(x+b)" is periodic with period “T”.
- "f(x) + a" is periodic with period “T”.
- "f(ax±b)" is periodic with a period “T/|a|”.

4: If “a” and “b” are non-zero real number and functions g(x) and h(x) are periodic functions having periods, “T_1” and “T_2” , then function $f(x) = ag(x) \pm bh(x)$ is also a periodic function. The period of f(x) is LCM of “T_1” and “T_2”.

Note: The fourth LCM rule is subject to certain restrictions. For complete detail read module titled “Periodic functions”.

Examples

Problem 2: Find the time period of the motion, whose displacement is given by :

$$x = \cos \omega t$$

Solution : We know that period of cosine function is “2π”. We also know that if “T” is the period of “f(t)”, then period of the function "f(at±b)" is “T/|a|”. Following this rule, time period of the given cosine function is :

$$\Rightarrow T = \frac{2\pi}{\omega}$$

Note that this expression is same that forms the basis of definition of angular frequency.

Problem 3: Find the time period of the motion, whose displacement is given by :

$$x = 2\cos(-3\pi t + 5)$$

Solution : We know that period of cosine function is “2π”. If “T” is the time period of f(t), then period of kf(t) is also “T”. Thus, coefficient “2” of trigonometric term has no effect on the period. We also know that if “T” is the period of “f(t)”, then period of the function "f(at±b)" is “T/|a|”. Following this rule, time period of the given cosine function is :

$$\Rightarrow T = \frac{2\pi}{3\pi} = \frac{2}{3}$$

Problem 4: Find the time period of the motion, whose displacement is given by :

$$x = \sin\omega t + \sin 2\omega t + \sin 3\omega t$$

Solution : We know that period of sine function is “2π”. We also know that if “T” is the period of “f(t)”, then period of the function "f(at±b)" is “T/|a|”. Following this rule, time periods of individual sine functions are :

$$\Rightarrow \text{Time period of ”sin ωt”}, T_1 = \frac{T}{\omega} = \frac{2\pi}{\omega}$$

$$\Rightarrow \text{Time period of ”sin 2ωt”}, T_2 = \frac{T}{2\omega} = \frac{2\pi}{2\omega} = \frac{\pi}{\omega}$$

$$\Rightarrow \text{Time period of ”sin 3ωt”}, T_3 = \frac{T}{3\omega} = \frac{2\pi}{3\omega} = \frac{2\pi}{3\omega}$$

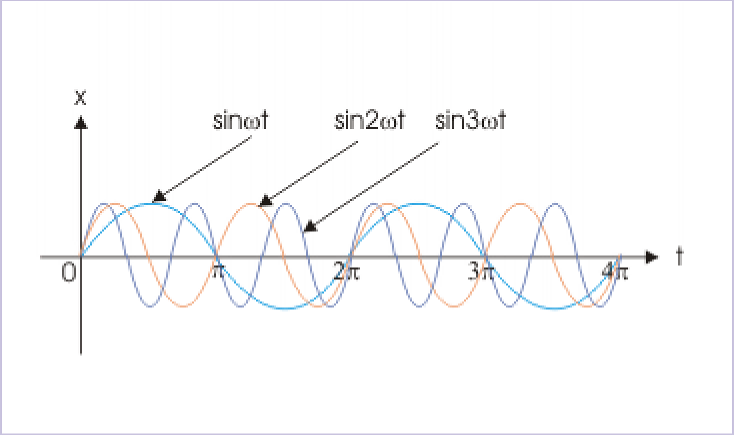
Applying LCM rule, we can find the period of combination. Now, LCM of fraction is obtained as :

$$\Rightarrow T = \frac{\text{LCM of numerators}}{\text{HCF of denominators}} = \frac{\text{LCM of “2π”, “π” and “2π”}}{\text{HCF of “ω”, “ω” and “3ω”}}$$

$$\Rightarrow T = \frac{2\pi}{\omega}$$

It is intuitive to understand that the frequencies of three functions are in the proportion 1:2:3. Their time periods are in inverse proportions 3:2:1. It means that by the time first function completes a cycle, second function completes two cycles and third function completes three cycles. This means that the period of first function encompasses the periods of remaining two functions. As such, time period of composite function is equal to time period of first function.

Period of function



The period of first function encompasses the periods of remaining two functions.

Simple harmonic motion

Oscillation is a periodic motion, which repeats after certain time interval. Simple harmonic motion is a special type of oscillation. In real time, all oscillatory motion dies out due to friction, if left unattended. We, therefore, need to replenish energy of the oscillatory motion to continue oscillating. However, we shall generally consider an ideal situation in which mechanical energy of the oscillating system is conserved. The object oscillates indefinitely. This is the reference case.

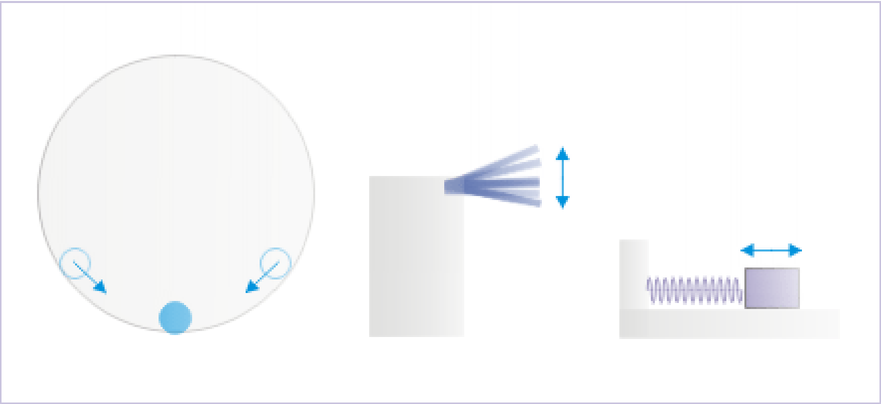
Though, we refer an object or body to describe oscillation, but it need not be. We can associate oscillation to energy, pattern and anything which varies about some value in a periodic manner. The oscillation, therefore, is a general concept. We shall, however, limit ourselves to physical oscillation, unless otherwise mentioned.

Further, study of oscillation has two distinct perspectives. One is the description of motion i.e. the kinematics of the motion. Second is the study of the cause of oscillation i.e. dynamics of the motion. In this module, we shall deal with the first perspective.

Oscillation

> Oscillation is a periodic, to and fro, bounded motion about a reference, usually the position of equilibrium.

Examples of oscillation



The object undergoes "to and fro" periodic motion.

The characteristics of oscillation are enumerated here :

- It is a periodic motion that repeats itself after certain time interval.
- The motion is about a point, which is often the position of equilibrium.
- The motion is bounded.

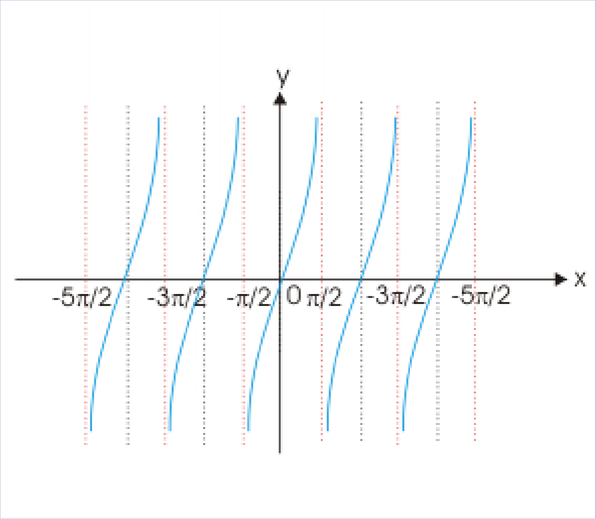
Note that revolution of second hand in the wrist watch is not an oscillation as the concept of “to and fro” motion about a point is missing. Thus, this is a periodic motion, but not an oscillatory motion. On the other hand, periodic swinging of pendulum in mechanical watch is an oscillatory motion.

Description of oscillation

We need a mathematical model to describe oscillation. We often use trigonometric functions. However, we can not use all of them. It is essentially because many of them are not bounded. Recall the plot of tangent function. It extends from minus infinity to plus infinity - periodically. Actually, only the sine and cosine trigonometric functions are bounded.

The plot of tangent function is shown here. Note that value of function extends from minus infinity to plus infinity.

Plot of tangent function

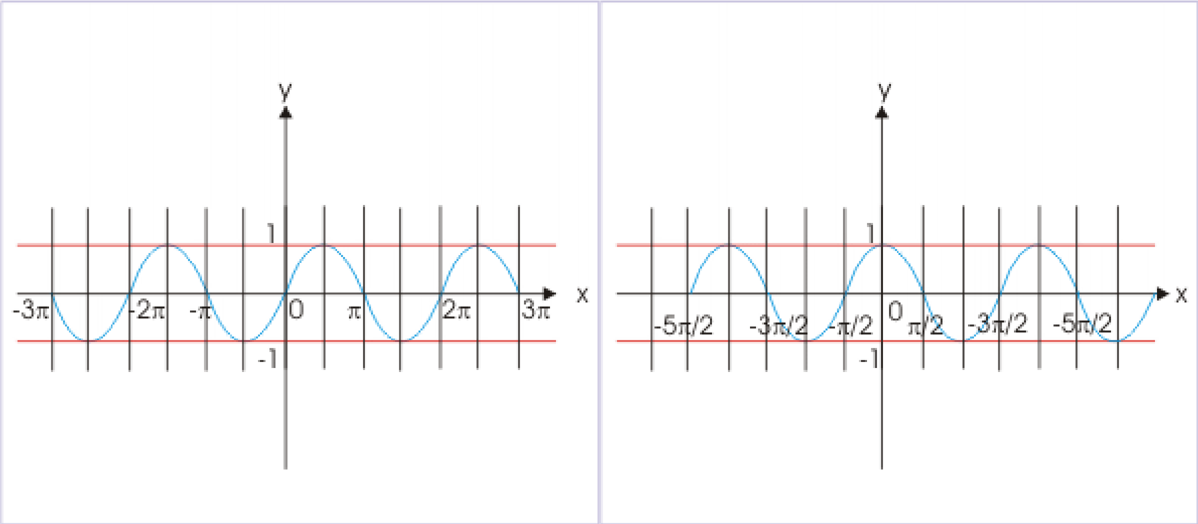


The function is not bounded.

The plots of sine and cosine functions are shown here. Note that value of function lies between "-1" and "1".

Plots of sine and cosine functions

The sine function is bounded. The cosine function is bounded.



Harmonic oscillation

Harmonic oscillation and simple harmonic oscillation both are described by a single bounded trigonometric function like sine or cosine function having single frequency (it is the number of times a motion is repeated in 1 second). The difference is only that simple harmonic function has constant amplitude over all time (amplitude represents maximum displacement from central or mean position of the periodic motion) as a result of which mechanical energy of the oscillating system is conserved.

Examples of harmonic motion are :

- $x = Ae^{-\omega t} \sin \omega t$

- $x = A\sin\omega t$
- $x = A\omega\cos\omega t$
- $x = A\sin\omega t + B\cos\omega t$

Of these, last three examples are simple harmonic oscillations.

Note that we can reduce fourth example, sum of two trigonometric functions, into a single trigonometric function with appropriate substitutions. As a matter of fact, we shall illustrate such reduction in appropriate context.

The simple harmonic oscillation is popularly known as simple harmonic motion (SHM). The important things to reemphasize here is that SHM denotes an oscillation, which does not involve change in amplitude. We shall learn that this represents a system in which energy is not dissipated. It means that mechanical energy of a system in SHM is conserved.

Non-harmonic oscillation

A non-harmonic oscillation is one, which is not harmonic motion. We can consider combination of two or more harmonic motions of different frequencies as an illustration of non-harmonic function.

$$x = A\sin\omega t + B\sin 2\omega t$$

We can not reduce this sum into a single trigonometric sine or cosine function and as such, motion described by the function is non-harmonic.

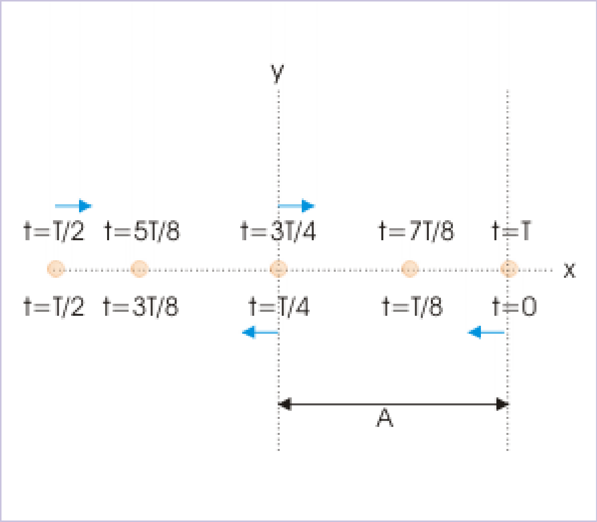
Simple harmonic motion

A simple harmonic motion can be conceived as a “to and fro” motion along an axis (say x-axis). In order to simplify the matter, we choose origin of the reference as the point about which particle oscillates. If we start our observation from positive extreme of the motion, then displacement of the particle “x” at a time “t” is given by :

$$x = A\cos\omega t$$

where “ω” is angular frequency and “t” is the time. The figure here shows the positions of the particle executing SHM at an interval of “T/8”. The important thing to note here is that displacements in different intervals are not equal, suggesting that velocity of the particle is not uniform. This also follows from the nature of cosine function. The values of cosine function are not equally spaced with respect to angles.

Simple harmonic motion

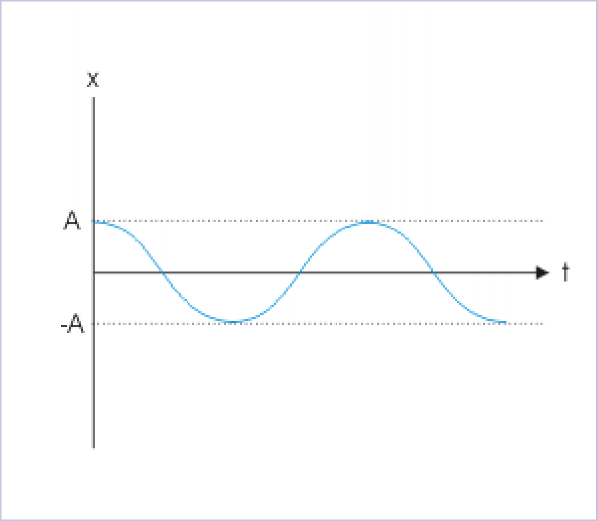


Positions of the particles at different times are shown.

Amplitude

We know that value of cosine function lies between “-1” and “1”. Hence, value of “x” varies between “-A” and “A”. If we plot the function describing displacement, then the plot is similar to that of cosine function except that its range of values lies between “-A” and “A”.

Amplitude



The scalar value of maximum displacement from the mean position is known as the amplitude of oscillation.

The value “A” denotes the maximum displacement in either direction. The scalar value of maximum displacement from the mean position is known as the amplitude of oscillation. If we consider pendulum, we can observe that farther is the point from which pendulum bob (within the permissible limit in which the bob executes SHM) is released, greater is the amplitude of oscillation. Similarly, greater is the stretch or compression in the spring executing SHM, greater is the amplitude. Alternatively, we can say that greater is the force causing motion, greater is the amplitude. In the nutshell, amplitude of SHM depends on the initial conditions of motion - force and displacement.

Time period

The time period is the time taken to complete one cycle of motion. In our consideration in which we have started observation from positive extreme, this is equal to time taken from start t =0 s to the time when the particle returns to the positive extreme position again.

$$\text{At} \quad t = 0, \quad \omega t = 0, \quad \cos\omega t = \cos 0^0 = 1$$

$$\Rightarrow x = A$$

$$\text{At} \quad t = \frac{2\pi}{\omega}, \quad \omega t = \omega X \frac{2\pi}{\omega} = 2\pi, \quad \cos\omega t = \cos 2\pi = 1,$$

$$\Rightarrow x = A$$

Thus, we see that particle takes a time “2π/ω” to return to the extreme position from which motion started. Hence, time period of SHM is :

$$\Rightarrow T = \frac{2\pi}{\omega} = \frac{2\pi}{2\pi\nu} = \frac{1}{\nu}$$

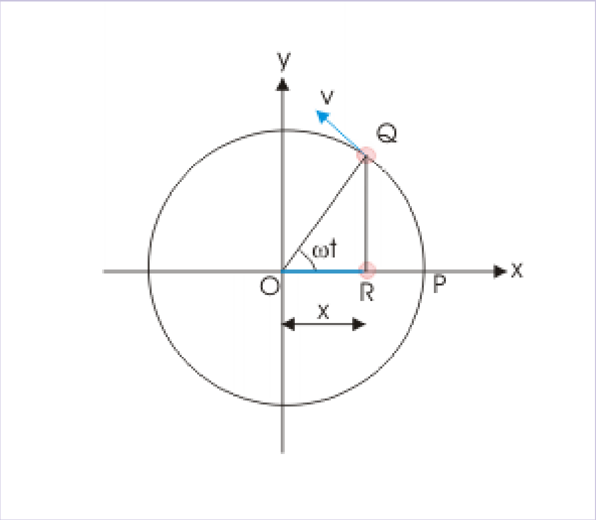
SHM and uniform circular motion

The expression for the linear displacement in “x” involves angular frequency :

$$x = A\cos\omega t$$

Clearly, we need to understand the meaning of angular frequency in the context of linear “to and fro” motion. We can understand the connection or the meaning of this angular quantity knowing that SHM can be interpreted in terms of uniform circular motion. Consider SHM of a particle along x-axis, while another particle moves along a circle at a uniform angular speed, "ω", in anticlockwise direction as shown in the figure. Let both particles begin moving from point “P” at t=0.

SHM and UCM



The displacement of particle in SHM is equal to projection of position of particle in UCM.

After time “t”, the particle executing uniform circular motion (UCM), covers an angular displacement “ωt” and reaches a point “Q” as shown in the figure. The projection of line joining origin, "O" and “Q” on x – axis is :

$$OR = x = A\cos\omega t$$

Now compare this expression with the expression of displacement of particle executing SHM. Clearly, both are same. It means projection of the position of the particle executing UCM is equal to the displacement of the particle executing SHM from the origin. This is the connection between two motions. Also, it is obvious that angular frequency “ω” is equal to the uniform angular speed, “ω" of the particle executing UCM .

In case, if this analogical interpretation does not help to interpret angular frequency, then we can simply think that angular frequency is product of “2π” and frequency “ν”.

$$\omega = 2\pi\nu$$

Phase constant

We used a cosine function to represent displacement of the particle in SHM. This function represents displacement for the case when we start observing motion of the particle at positive extreme. At t = 0,

$$x = A\cos\omega t = A\cos 0^0 = A$$

What if we want to observe motion from the position when the particle is at mean position i.e. at "O". We know that sine of zero is zero. Knowing the nature of sine curve, we can ,intuitively, say that sine function would fit in the requirement in this case and displacement is given as :

$$x = A\sin\omega t$$

However, if we want to stick with the cosine function, then there is a way around. We know that :

$$\cos\theta = \sin\left(\theta \pm \frac{\pi}{2}\right)$$

Keeping this in mind, we represent the displacement as :

$$\Rightarrow x = A\cos\left(\omega t \pm \frac{\pi}{2}\right)$$

Let us check out the position of the particle at t = 0,

$$\Rightarrow x = A\cos\left(\pm\frac{\pi}{2}\right) = 0$$

Clearly, cosine function represents displacement with this modification even in case when we start observing motion of the particle at mean position. In other words, cosine function, as modified, is equivalent to sine function.

We see that adding or subtracting "π/2" serves the purpose. When we add the angle "π/2", the position of particle, at mean position, is ahead of positive extreme. The particle has moved from the positive extreme to the mean position. When we subtract the angle "-π/2", the particle, at mean position, lags behind the position at positive extreme. In other words, the particle has moved from the negative extreme to the mean position.

Here, we note that “ωt” is dimensionless angle and is compatible with the angle being added or subtracted :

$$\left[\omega T\right] = \left[\frac{2\pi}{T} XT\right] = \left[2\pi\right] = \text{dimensionless}$$

Representation of displacement from positions other than extreme position in this manner gives rise to an important concept of “phase constant” . The angle being added or subtracted to represent change in start position is also known as "phase constant" or “phase angle” or “initial phase” or “epoch”. This concept allows us to represent displacement whatever be the initial condition (position and direction of velocity – whether particle is moving towards the positive extreme (negative phase constant) or moving away from the positive extreme (positive phase constant). For an intermediate position, we can write displacement as :

$$\Rightarrow x = A\cos(\omega t + \varphi)$$

Note that we have purposely removed negative sign as we can alternatively say that phase constant has positive or negative value, depending on its state of motion at t = 0. The concept of phase constant will be more clearer if we study the plots of the motion for phase "0", “φ” and “-φ” as illustrated in the figure below.

Phase constant

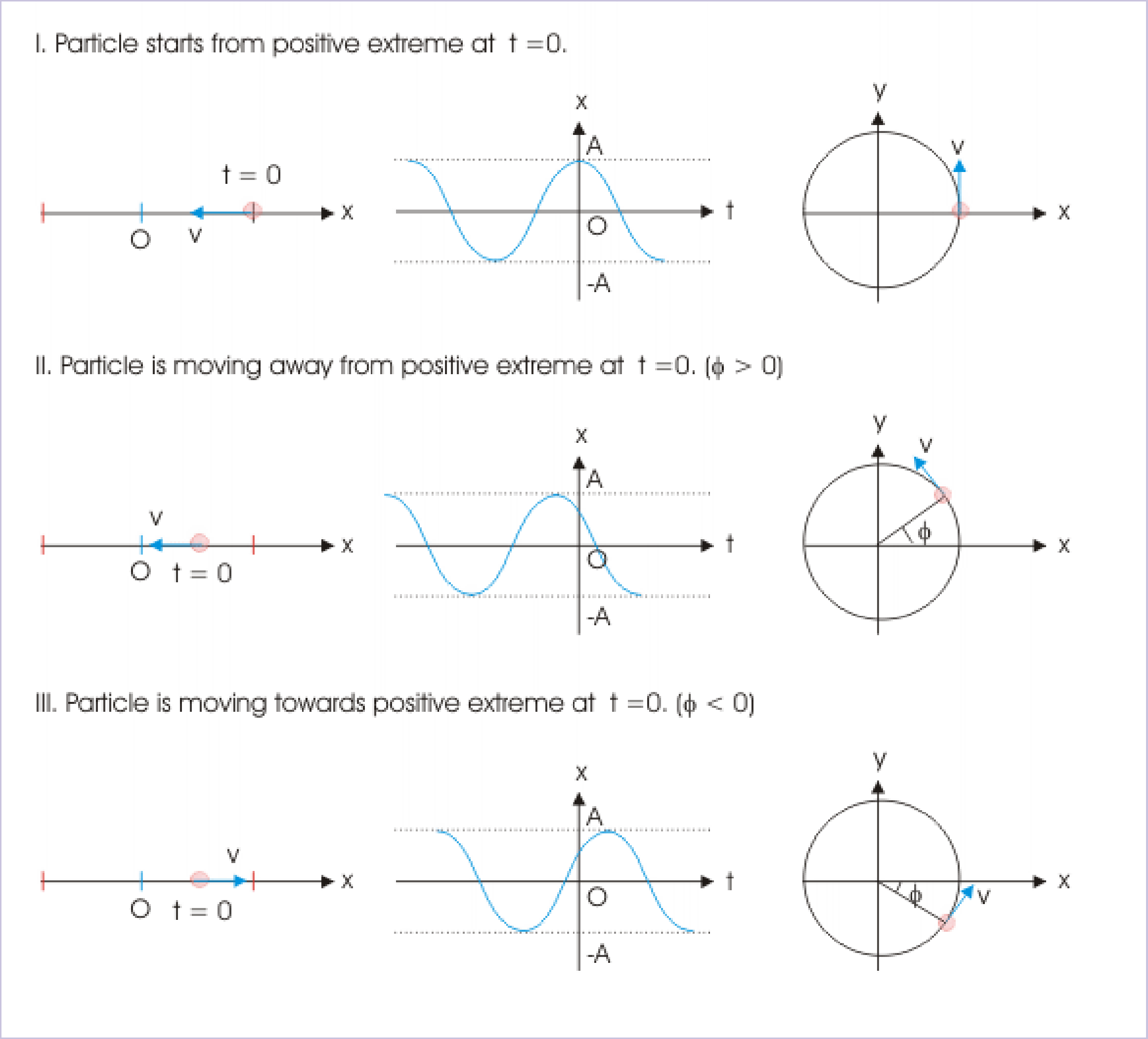

I. Particle starts from positive extreme at t =0.
t = 0
O
v
x
x
A
O
-A
t
y
v
x
II. Particle is moving away from positive extreme at t =0. (ϕ > 0)
v
O
t = 0
x
x
A
O
-A
t
y
v
ϕ
x
III. Particle is moving towards positive extreme at t =0. (ϕ < 0)
v
O
t = 0
x
x
A
O
-A
t
y
ϕ
v
x

Illustration of different phase constants

The figure here captures the meaning of phase constant. Let us begin with the uppermost row of figures. We start observing motion from positive extreme (left figure), phase constant is zero. The displacement is maximum “A”. The particle is moving from positive extreme position to negative extreme (middle figure). The equivalent particle, executing uniform circular motion, is at positive extreme (right figure).

In the middle row of the figures, we start observing motion, when the particle is between positive extreme and mean position (left figure), but moving away from the positive extreme. Here, phase constant is positive. The displacement is not equal to amplitude. Actually, maximum displacement event is already over, when we start observation (middle figure). The particle is moving from its position to negative extreme. The equivalent particle, executing uniform circular motion, is at an angle “φ” ahead from the positive extreme position (right figure).

In the lowermost row of the figures, we start observing motion, when the particle is between positive extreme and mean position (left figure), but moving towards the positive extreme. Here, phase constant is negative. The displacement is not equal to amplitude. Actually, maximum displacement event is yet to be realized, when we start observation (middle figure). The particle is moving from its position to the positive extreme position. The equivalent particle, executing uniform circular motion, is at an angle “φ” behind from the positive extreme position (right figure).

From the description as above, we conclude that phase constant depends on initial two attributes of the particle in motion (i) its position and (ii) its velocity (its direction of motion).

From the discussion, it is also clear that using either "cosine" or "sine" function is matter of choice. Both functions can equivalently be used to describe SHM with appropriate phase constant.

Phase

Simply put phase is the argument (angle) of trigonometric function used to represent displacement.

$$x = A\cos(\omega t + \varphi)$$

The argument “ωt + φ” is the phase of the SHM. Clearly phase is an angle like “π/3” or “π/6”. Sometimes, we loosely refer phase in terms of time period like “T/4”, we need to convert the same into equivalent angle before using in the relation.

The important aspect of phase is that if we know phase of a SHM, we know a whole lot of things about SHM. By evaluating expression of phase, we know (i) initial position (ii) direction of motion (iii) frequency and angular frequency (iv) time period and (v) phase constant.

Consider a SHM equation (use SI units) :

$$x = \sin\left(\frac{\pi t}{3} - \frac{\pi}{6}\right)$$

Clearly,

At t = 0,

$$\text{Initial position}, x_0 = \sin\left(\frac{\pi}{3}X0 - \frac{\pi}{6}\right) = \sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2} = -0.5 \quad m$$

Further,

$$\text{Amplitude}, A = 1 \quad m$$

$$\text{Angular frequency}, \omega = \frac{\pi}{3}$$

$$\text{Time period}, T = \frac{2\pi}{\omega} = \frac{2\pi}{\frac{\pi}{3}} = 6 \quad s$$

$$\text{Phase constant}, \varphi = -\frac{\pi}{6} \quad \text{radian}$$

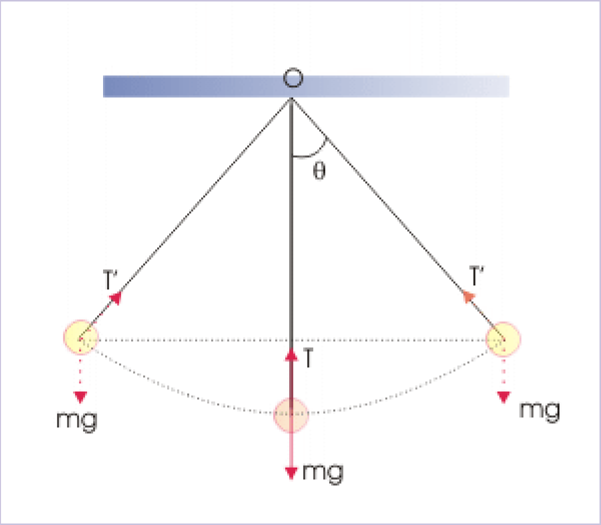
As phase constant is negative, the particle is moving towards positive extreme position.

SHM equation

Force is the cause of simple harmonic motion. However, it plays a very different role than that in translational or rotational motion. First thing that we should be aware that oscillating systems like block-spring arrangement or a pendulum are examples of systems in stable equilibrium. When we apply external force, it tends to destabilize or disturb the state of equilibrium by imparting acceleration to the object in accordance with Newton’s second law.

Secondly, role of destabilizing external force is one time act. The oscillation does not require destabilizing external force subsequently. It, however, does not mean that SHM is a non-accelerated motion. As a matter of fact, oscillating system generates a restoring mechanism or restoring force that takes over the role of external force once disturbed. The name “restoring” signifies that force on the oscillating object attempts to restore equilibrium (central position in the figure) – undoubtedly without success in SHM.

System in stable equilibrium



Restoring force(s) tends to restore equilibrium.

SHM is an accelerated motion in which object keeps changing its velocity all the time. The analysis of SHM involves consideration of “restoring force” – not the external force that initially starts the motion. Further, we need to understand that initial external force and hence restoring force are relatively small than the force required to cause translation or rotation. For example, if we displace pendulum bob by a large angle and release the same for oscillation, then the force on the system may not fulfill the requirement of SHM and as such the resulting motion may not be a SHM.

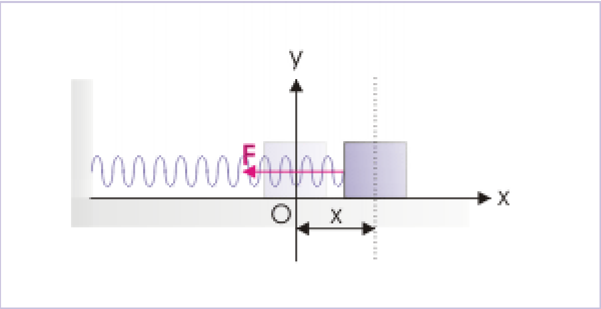
We conclude the discussion by enumerating requirements of SHM as :

- The object is in stable equilibrium before start of the motion.
- External destabilizing force is applied only once.
- The object accelerates and executes SHM under the action of restoring force.
- The magnitude of restoring force or displacement is relatively small.
- There is no dissipation of energy during motion (ideal reference assumption).

Force equation

Here, we set out to figure out nature of restoring force that maintains SHM. For understanding purpose we consider the block-spring system and analyze “to and fro” motion of the block. Let the origin of reference coincides with the position of the block for the neutral length of spring. The block is moved right by a small displacement “x” and released to oscillate about neutral position or center of oscillation. The restoring spring force is given by (“k” is spring constant) :

Block-spring system



Restoring force(s) tends to restore equilibrium.

$$F=-kx$$

In the case of pendulum, we describe motion in terms of torque as it involves angular motion. Here, torque is (we shall study this relation later):

$$\tau=-mgl\theta$$

In either case, we see that “cause” (whether force or torque) is proportional to negative of displacement – linear or angular as the case be. Alternatively, we can also state the nature of restoring force in terms of acceleration,

Linear acceleration, $a=-\frac{k}{m}x$ ------ for linear SHM of block-spring system

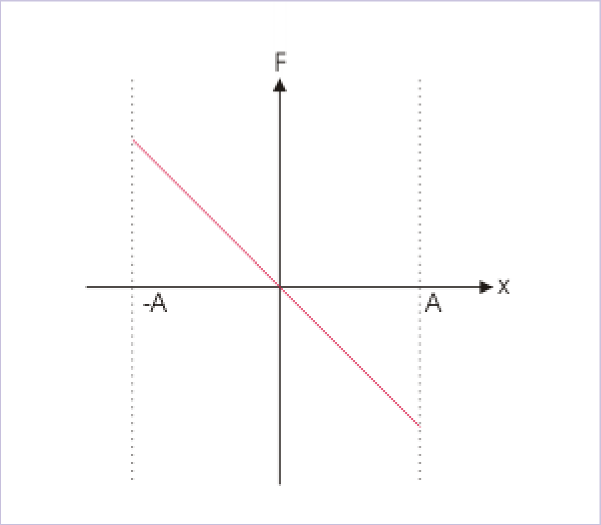
Angular acceleration, $\alpha=-\frac{mgl}{I}\theta$ ------ for rotational SHM of pendulum

where “m” and “I” are mass and moment of inertia of the oscillating objects in two systems.

In order to understand the nature of cause, we focus on the block-spring system. When the block is to the right of origin, “x” is positive and restoring spring force is negative. This means that restoring force (resulting from elongation of the spring) is directed left towards the neutral position (center of oscillation). This force accelerates the block towards the center. As a result, the block picks up velocity till it reaches the center.

The plot, here, depicts nature of force about the center of oscillation bounded between maximum displacements on either side.

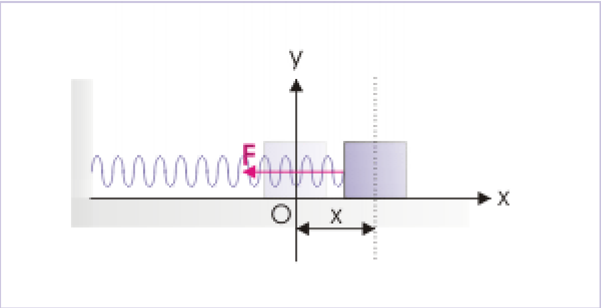
Nature of restoring force



Restoring force(s) tends to
restore equilibrium.

As the block moves past the center, “x” is negative and force is positive. This means that restoring force (resulting from compression of the spring) is directed right towards the center. The acceleration is positive, but opposite to direction of velocity. As such restoring force decelerates the block.

Block-spring system



Restoring force(s) tends to
restore equilibrium.

In the nutshell, after the block is released at one extreme, it moves, first, with acceleration up to the center and then moves beyond center towards left with deceleration till velocity becomes zero at the opposite extreme. It is clear that block has maximum velocity at the center and least at the extreme positions (zero).

From the discussion, the characterizing aspects of the restoring force responsible for SHM are :

- The restoring force is always directed towards the center of oscillation.
- The restoring force changes direction across the center.
- The restoring force first accelerates the object till it reaches the center and then decelerates the object till it reaches the other extreme.
- The process of “acceleration” and “deceleration” keeps alternating in each half of the motion.

Differential form of SHM equation

We observed that acceleration of the object in SHM is proportional to negative of displacement. Here, we shall formulate the general equation for SHM in linear motion with the understanding that same can be extended to SHM along curved path. In that case, we only need to replace linear quantities with corresponding angular quantities.

$$a = -\omega^2 x$$

where “ω^2” is a constant. The constant “ω” turns out to be angular frequency of SHM. This equation is the basic equation for SHM. For block-spring system, it can be seen that :

$$\omega = \sqrt{\left(\frac{k}{m}\right)}$$

where “k” is the spring constant and “m” is the mass of the oscillating block. We can, now, write acceleration as differential,

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

$$\Rightarrow \frac{d^2x}{dt^2} + \omega^2 x = 0$$

This is the SHM equation in differential form for linear oscillation. A corresponding equation of motion in the context of angular SHM is :

$$\Rightarrow \frac{d^2\theta}{dt^2} + \omega^2 \theta = 0$$

where "θ" is the angular displacement.

Solution of SHM differential equation

In order to solve the differential equation, we consider position of the oscillating object at an initial displacement x_0 at t =0. We need to emphasize that “x_0” is initial position – not the extreme position, which is equal to amplitude “A”. Let

$$t = 0, \quad x = x_{0,} \quad v = v_0$$

We shall solve this equation in two parts. We shall first solve equation of motion for the velocity as acceleration can be written as differentiation of velocity w.r.t to time. Once, we have the expression for velocity, we can solve velocity equation to obtain a relation for displacement as its derivative w.r.t time is equal to velocity.

Velocity

We write SHM equation as differential of velocity :

$$a = \frac{dv}{dt} == -\omega^2 x$$

$$\Rightarrow \frac{\mathbb{d}v}{\mathbb{d}x} X \frac{\mathbb{d}x}{\mathbb{d}t} = -\omega^2 x$$

$$\Rightarrow v\frac{\mathbb{d}v}{\mathbb{d}x} = -\omega^2 x$$

Arranging terms with same variable on either side, we have :

$$\Rightarrow v\mathbb{d}v = -\omega^2 x\mathbb{d}x$$

Integrating on either side between interval, while keeping constant out of the integral sign :

$$\Rightarrow \int\limits_{v_0}^{v} v\mathbb{d}v = -\omega^2 \int\limits_{x_0}^{x} x\mathbb{d}x$$

$$\Rightarrow \left[\frac{v^2}{2}\right]_{v_0}^{v} = -\omega^2 \left[\frac{x^2}{2}\right]_{x_0}^{x}$$

$$\Rightarrow \left(v^2 - v_0^2\right) = -\omega^2\left(x^2 - x_0^2\right)$$

$$\Rightarrow v = \sqrt{\left\{\left(v_0^2 + \omega^2 x_0^2\right) - \omega^2 x^2\right\}}$$

$$\Rightarrow v = \omega\sqrt{\left\{\left(\frac{v_0^2}{\omega^2} + x_0^2\right) - x^2\right\}}$$

We put $\left(\frac{v_0^2}{\omega^2} + x_0^2\right) = A^2$. We shall see that “A” turns out to be the amplitude of SHM.

$$\Rightarrow v = \omega\sqrt{\left(A^2 - x^2\right)}$$

This is the equation of velocity. When x = A or -A,

$$\Rightarrow v = \omega\sqrt{(A^2 - A^2)} = 0$$

when x = 0,

$$\Rightarrow v\max = \omega\sqrt{(A^2 - 0^2)} = \omega A$$

Displacement

We write velocity as differential of displacement :

$$v = \frac{dx}{dt} = \omega\sqrt{(A^2 - x^2)}$$

Arranging terms with same variable on either side, we have :

$$\Rightarrow \frac{dx}{\sqrt{(A^2 - x^2)}} = \omega dt$$

Integrating on either side between interval, while keeping constant out of the integral sign :

$$\Rightarrow \int_{x_0}^{x} \frac{dx}{\sqrt{(A^2 - x^2)}} = \omega \int_{0}^{t} dt$$

$$\Rightarrow \left[\sin^{-1}\frac{x}{A}\right]_{x_0}^{x} = \omega t$$

$$\Rightarrow \sin^{-1}\frac{x}{A} - \sin^{-1}\frac{x_0}{A} = \omega t$$

Let $\sin^{-1}\frac{x_0}{A} = \varphi$. We shall see that “φ” turns out to be the phase constant of SHM.

$$\Rightarrow \sin^{-1}\frac{x}{A} = \omega t + \varphi$$

$$\Rightarrow x = A\sin(\omega t + \varphi)$$

This is one of solutions of the differential equation. We can check this by differentiating this equation twice with respect to time to yield equation of motion :

$$\frac{dx}{dt} = A\omega\cos(\omega t + \varphi)$$

$$\frac{d^2x}{dt^2} = -A\omega^2\sin(\omega t + \varphi) = -\omega^2 x$$

$$\Rightarrow \frac{d^2x}{dt^2} + \omega^2 x = 0$$

Similarly, it is found that equation of displacement in cosine form, $x = A\cos(\omega t + \varphi)$, also satisfies the equation of motion. As such, we can use either of two forms to represent displacement in SHM. Further, we can write general solution of the equation as :

$$x = A\sin\omega t + B\cos\omega t$$

This equation can be reduced to single sine or cosine function with appropriate substitution.

Example

Problem 1: Find the time taken by a particle executing SHM in going from mean position to half the amplitude. The time period of oscillation is 2 s.

Solution : Employing expression for displacement, we have :

$$x = A\sin\omega t$$

We have deliberately used sine function to represent displacement as we are required to determine time for displacement from mean position to a certain point. We could ofcourse stick with cosine function, but then we would need to add a phase constant “π/2” or “-π/2”. The two approach yields the same expression of displacement as above.

Now, according to question,

$$\Rightarrow \frac{A}{2} = A \sin \omega t$$

$$\Rightarrow \sin \omega t = \frac{1}{2} = \sin \frac{\pi}{6}$$

$$\Rightarrow \omega t = \frac{\pi}{6}$$

$$\Rightarrow t = \frac{\pi}{6\omega} = \frac{\pi T}{6 X 2\pi} = \frac{T}{12} = \frac{2}{12} = \frac{1}{6} \quad s$$

Linear SHM

The subject matter of this module is linear SHM – harmonic motion along a straight line about the point of oscillation. There are various physical quantities associated with simple harmonic motion. Here, we intend to have a closer look at quantities associated with SHM like velocity, acceleration, work done, kinetic energy, potential energy and mechanical energy etc. For the sake of completeness, we shall also have a recap of concepts already discussed in earlier modules.

The SHM force relation “F = -kx” is a generic form of equation for linear SHM – not specific to block-spring system. In the case of block-spring system, “k” is the spring constant. This point is clarified to emphasize that relations that we shall be developing in this module applies to all linear SHM and not to a specific case.

Since displacement of SHM can be represented either in cosine or sine forms, depending where we start observing motion at t = 0. For someone, it is easier to visualize beginning of SHM, when particle is released from positive extreme. On the other hand, expression in sine form is convenient as particle is at the center of oscillation at t = 0. For this reason, some prefer sine representation.

The very fact that there are two ways to represent displacement may pose certain ambiguity or uncertainty in mind. We shall , therefore, strive to maintain complete independence of forms with the understanding that when it is cosine function, then starting reference is positive extreme and if it is sine function, then starting reference is center of oscillation. In order to illustrate flexibility, we shall be using “sine” expression of displacement in this module instead of cosine function, which has so far been used.

Displacement

The displacement of the particle is given by :

$$x = A\sin(\omega t + \varphi)$$

where “A” is the amplitude,"ω" is angular frequency, “φ” is the phase constant and “ωt + φ” is the phase. Clearly, displacement is periodic with respect to time as it is represented by bounded trigonometric function. The displacement “x” varies between “-A” and “A”.

Velocity

The velocity of the particle as obtained from the solution of SHM equation is given by :

$$v = \omega\sqrt{(A^2 - x^2)}$$

This is the relation of velocity of the particle with respect to displacement along the path of oscillation, bounded between “-ωA” and “ωA”. We can obtain a relation of velocity with respect to time by substituting expression of displacement “x” in the above equation :

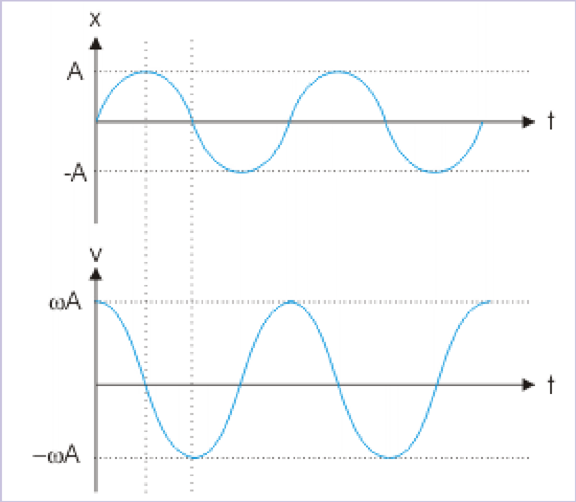
$$\Rightarrow v = \omega\sqrt{(A^2 - x^2)} = \omega\sqrt{\{A^2 - A^2 \sin^2(\omega t + \varphi)\}} = \omega A \cos(\omega t + \varphi)$$

We can ,alternatively, deduce this expression by differentiating displacement, “x”, with respect to time :

$$\Rightarrow v = \frac{dx}{dt} = \frac{d}{dt} A \sin(\omega t + \varphi) = \omega A \cos(\omega t + \varphi)$$

The variation of velocity with respect to time is sinusoidal and hence periodic. Here, we draw both displacement and velocity plots with respect to time in order to compare how velocity varies as particle is at different positions.

Velocity - time plot



The velocity is represented by cosine function.

The upper figure is displacement – time plot, whereas lower figure is velocity – time plot. We observe following important points about variation of velocity :

- If displacement is sine function, then velocity function is cosine function and vice-versa.
- The range of velocity lies between "-ωA" and "ωA".
- The velocity attains maximum value two times in a cycle at the center – (i) moving from negative to positive extreme and then (ii) moving from positive to negative extreme.
- The velocity at extreme positions is zero.

Acceleration

The acceleration in linear motion is given as :

$$a = -\omega^2 x = -\frac{k}{m}x$$

Substituting for displacement "x", we get an expression in variable time, "t" :

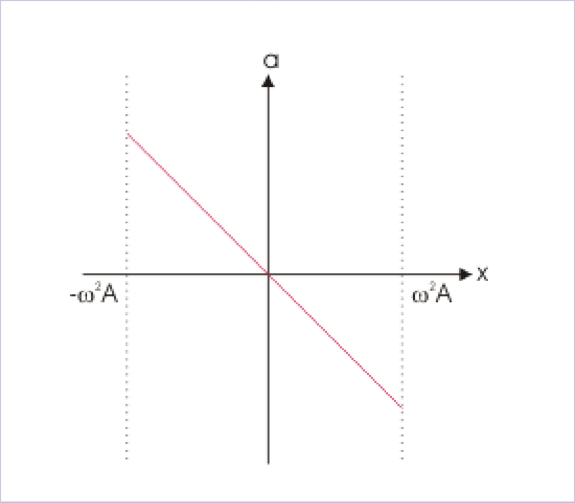
$$\Rightarrow a = -\omega^2 x = -\omega^2 A \sin(\omega t + \varphi)$$

We can obtain this relation also by differentiating displacement function twice or by differentiating velocity function once with respect to time. Few important points about the nature of acceleration should be kept in mind :

1: Acceleration changes its direction about point of oscillation. It is always directed towards the center whatever be the position of the particle executing SHM.

2: Acceleration linearly varies with negative of displacement. We have seen that force-displacement plot is a straight line. Hence, acceleration – displacement plot is also a straight line. It is positive when “x” is negative and it is negative when “x” is positive.

Acceleration - displacement plot



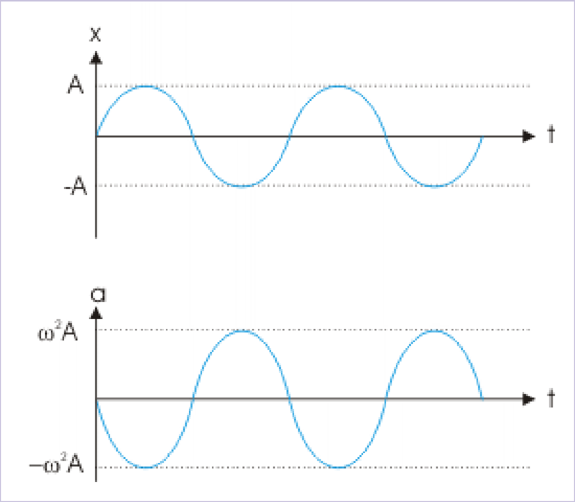
The acceleration - displacement is a straight line bounded between two values.

3: Nature of force with respect to time, however, is not linear. If we combine the expression of acceleration and displacement, then we have :

$$\Rightarrow a = -\omega^2 x = -\omega^2 A \sin(\omega t + \varphi)$$

Here, we draw both displacement and acceleration plots with respect to time in order to compare how acceleration varies as particle is at different positions.

Acceleration - time plot



The acceleration is represented by negative sine function.

The upper figure is displacement – time plot, whereas lower figure is acceleration – time plot. We observe following important points about variation of acceleration :

- If displacement is sine function, then acceleration function is also sine function, but with a negative sign.
- The range of acceleration lies between “$-\omega^2 A$” and “$\omega^2 A$”.
- The acceleration attains maximum value at the extremes.
- The acceleration at the center is zero.

4: Since force is equal to product of mass and acceleration, F = ma, it is imperative that nature of force is similar to that of acceleration. It is given by :

$$\Rightarrow F = ma = -kx = -m\omega^2 x = -m\omega^2 A \sin(\omega t + \varphi)$$

Frequency, angular frequency and time period

The angular frequency is given by :

$$\omega = \sqrt{\left(\frac{k}{m}\right)} = \sqrt{\left|\frac{a}{x}\right|} = \sqrt{\left|\frac{\text{acceleration}}{\text{displacement}}\right|}$$

We have used the fact "$F = ma = -kx$" to write different relations as above.

Time period is obtained from the defining relation :

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\left(\frac{m}{k}\right)} = 2\pi\sqrt{\left|\frac{x}{a}\right|} = \sqrt{\left|\frac{\text{displacement}}{\text{acceleration}}\right|}$$

Frequency is obtained from the defining relation :

$$\nu = \frac{1}{T} = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\left(\frac{k}{m}\right)} = \frac{1}{2\pi}\sqrt{\left|\frac{a}{x}\right|} = \sqrt{\left|\frac{\text{acceleration}}{\text{displacement}}\right|}$$

Kinetic energy

The instantaneous kinetic energy of oscillating particle is obtained from the defining equation of kinetic energy as :

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2\left(A^2 - x^2\right) = \frac{1}{2}k\left(A^2 - x^2\right)$$

The maximum value of KE corresponds to position when speed has maximum value. At x = 0,

$$\Rightarrow K_{\text{max}} = \frac{1}{2}m\omega^2\left(A^2 - 0^2\right) = \frac{1}{2}kA^2$$

The minimum value of KE corresponds to position when speed has minimum value. At x = A,

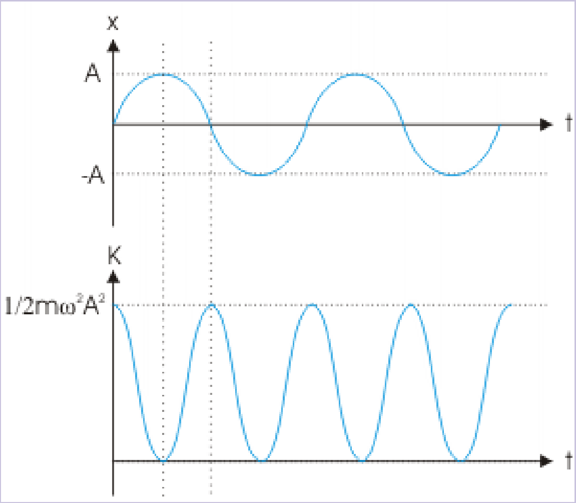
$$\Rightarrow K_{\text{min}} = \frac{1}{2}m\omega^2\left(A^2 - A^2\right) = 0$$

By substituting for “x” in the equation of kinetic energy, we get expression of kinetic energy in terms of variable time, “t” as :

$$K = \frac{1}{2} m\omega^2 A^2 \cos^2(\omega t + \varphi)$$

The kinetic energy – time plot is shown in the figure. We observe following important points about variation of kinetic energy :

Kinetic energy - time plot



The kinetic energy is represented by squared cosine function.

1: The KE function is square of cosine function. It means that KE is always positive.

2: The time period of KE is half that of displacement. We know the trigonometric identity :

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

Applying this trigonometric identity to the square of cosine term in the expression of kinetic energy as :

$$\Rightarrow K = \frac{1}{4} m\omega^2 A^2 \left\{1 + \cos 2(\omega t + \varphi)\right\} = \frac{1}{4} m\omega^2 A^2 \left\{1 + \cos(2\omega t + 2\varphi)\right\}$$

Applying rules for finding time period, we know that period of function “kf(x)” is same as that of “f(x)”. Hence, period of “K” is same as that of “1 + cos (2ωt + 2φ)”. Also, we know that period of function “f(x) + a” is same as that of “f(x)”. Hence, period of “K” is same as that of “cos (2ωt + 2φ)”. Now, period of “f(ax±b)” is equal to period of “f(x)” divided by “|a|”. Hence, period of “K” is :

$$\Rightarrow \text{Period} = \frac{2\pi}{2\omega} = \frac{\pi}{\omega} = \frac{T}{2}$$

As time period of variation of kinetic energy is half, the frequency of “K” is twice that of displacement. For this reason, kinetic energy – time plot is denser than that of displacement – time plot.

Potential energy

We recall that potential energy is an attribute of conservative force system. The first question that we need to answer is whether restoring force in SHM is a conservative force? One of the assumptions, which we made in the beginning, is that there is no dissipation of energy in SHM. It follows, then, that restoring force in SHM is a conservative force.

Second important point that we need to address is to determine a reference zero potential energy. We observe that force on the particle in SHM is zero at the center and as such serves to become the zero reference potential energy. Now, potential energy at a position “x” is equal to negative of the work done in taking the particle from reference point to position “x”.

$$U = -W = -\int F dx = -\int (-kx) dx = k \int x dx$$

Integrating in the interval, we have :

$$\Rightarrow U = k \int_0^x x dx = k \left[\frac{x^2}{2} \right]_0^x = \frac{1}{2} k x^2$$

Thus, instantaneous potential energy of oscillating particle is given as :

$$\Rightarrow U = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2x^2$$

The maximum value of PE corresponds to position when speed is zero. At x = A,

$$\Rightarrow U_{\max} = \frac{1}{2}kA^2 = \frac{1}{2}m\omega^2A^2$$

The minimum value of PE corresponds to position when speed has maximum value. At x = 0,

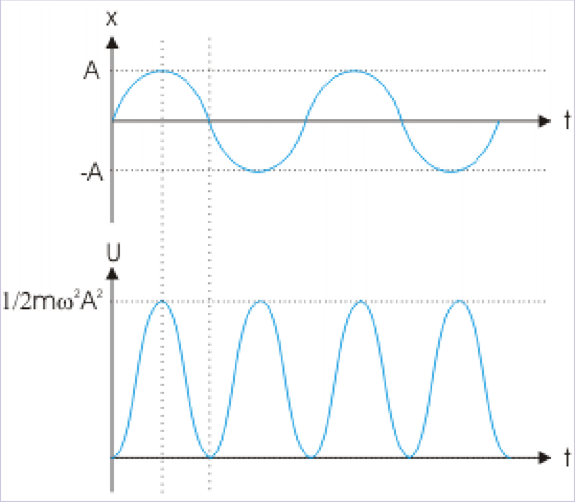
$$\Rightarrow U_{\min} = \frac{1}{2}kX0^2 = 0$$

By substituting for “x” in the equation of potential energy, we get expression of kinetic energy in terms of variable time, “t” as :

$$U = \frac{1}{2}kA^2\sin^2(\omega t + \varphi) = \frac{1}{2}m\omega^2A^2\sin^2(\omega t + \varphi)$$

The potential energy – time plot is shown in the figure. We observe following important points about variation of kinetic energy :

Potential energy - time plot



The kinetic energy is represented by squared sine function.

1: The KE function is square of sine function. It means that PE is always positive.

2: The time period of KE is half that of displacement. We have already proved the same in the case of kinetic energy. We can extend the reason in the case of potential energy as well :

$$\Rightarrow \text{Period} = \frac{2\pi}{2\omega} = \frac{\pi}{\omega} = \frac{T}{2}$$

As time period of variation is half, the frequency of “U” is twice that of displacement. For this reason, potential energy – time plot is denser than that of displacement – time plot.

Mechanical energy

The basic requirement of SHM is that mechanical energy of the system is conserved. At any point or at any time of instant, the sum of potential and kinetic energy of the system in SHM is constant. This is substantiated by evaluating sum of two energies :

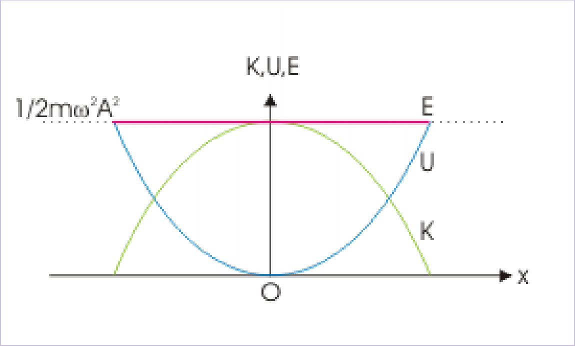
$$E = K + U$$

Using expressions involving displacement, we have :

$$\Rightarrow E = \frac{1}{2}m\omega^2\left(A^2 - x^2\right) + \frac{1}{2}m\omega^2 x^2 = \frac{1}{2}m\omega^2 A^2$$

The plots of kinetic, potential and mechanical energy with respect to displacement are drawn in the figure. Note that the sum of kinetic and potential energy is always a constant, which is equal to the mechanical energy of the particle in SHM.

Mechanical energy - displacement plot



The sum of potential and kinetic energy is a constant.

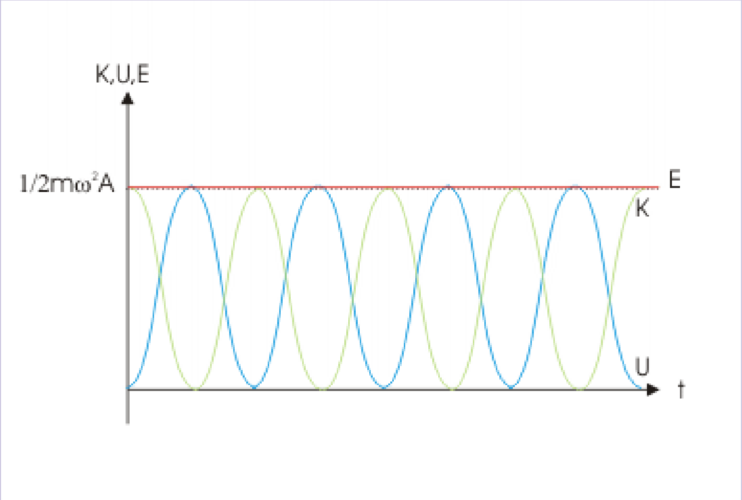
We can also obtain expression of mechanical energy, using time dependent expressions of kinetic and potential energy as :

$$\Rightarrow E=\frac{1}{2}m\omega^2A^2\cos^2(\omega t+\varphi)+\frac{1}{2}m\omega^2A^2\sin^2(\omega t+\varphi)$$

$$\Rightarrow E=\frac{1}{2}m\omega^2A^2\left\{\cos^2(\omega t+\varphi)+\sin^2(\omega t+\varphi)\right\}=\frac{1}{2}m\omega^2A^2$$

The mechanical energy – time plot is shown in the figure. We observe following important points about variation of energy with respect to time :

Mechanical energy - time plot



The sum of potential and kinetic energy is
a constant.

- Mechanical energy – time plot is a straight line parallel to time axis. This signifies that mechanical energy of particle in SHM is conserved.
- There is transformation of energy between kinetic and potential energy during SHM.
- At any instant, the sum of kinetic and potential energy is equal to $\frac{1}{2}m\omega^2A^2$ or $\frac{1}{2}kA^2$, which is equal to maximum values of either kinetic or potential energy.

Example

Problem 1: The potential energy of an oscillating particle of mass “m” along straight line is given as :

$$U(x) = a + b(x - c)^2$$

The mechanical energy of the oscillating particle is “E”.

1. Determine whether oscillation is SHM?
2. If oscillation is SHM, then find amplitude and maximum kinetic energy.

Solution : If the motion is SHM, then restoring force is a conservative force. The potential energy is, then, defined such that :

$$dU = -F dx$$

$$\Rightarrow F = -\frac{dU}{dx} = -2b(x - c)$$

In order to find the center of oscillation, we put F = 0.

$$F = -2b(x - c) = 0 \quad \Rightarrow x - c = 0 \quad \Rightarrow x = c$$

This means that particle is oscillating about point x = c. The displacement of the particle in that case is “x-c” – not “x”. This, in turn, means that force is proportional to negative of displacement, “x-c”. Hence, particle is executing SHM.

Alternatively, put y = x-c :

$$F = -2by$$

This means that particle is executing SHM about y = 0. This means x-c = 0, which in turn, means that particle is executing SHM about x = c.

The mechanical energy is related to amplitude by the relation :

$$E = \frac{1}{2}m\omega^2 A^2$$

$$\Rightarrow A = \sqrt{\left(\frac{2E}{m\omega^2}\right)}$$

Now, $m\omega^2 = k = 2b$. Hence,

$$\Rightarrow A = \sqrt{\left(\frac{2E}{2b}\right)} = \sqrt{\left(\frac{E}{b}\right)}$$

The potential energy is minimum at the center of oscillation i.e. when x = c. Putting this value in the expression of potential energy, we have :

$$\Rightarrow U_{\text{min}} = a + b(c - c)^2 = a$$

It is important to note that minimum value of potential energy need not be zero. Now, kinetic energy is maximum, when potential energy is minimum. Hence,

$$K_{\text{max}} = E - U_{\text{min}} = E - a$$

Angular SHM

Angular SHM involves “to and fro” angular oscillation of a body about a central position or orientation. The particle or the body undergoes small angular displacement about mean position. This results, when the body under stable equilibrium is disturbed by a small external torque. In turn, the rotating system generates a restoring torque, which tries to restore equilibrium.

Learning about angular SHM is easy as there runs a parallel set of governing equations for different physical quantities involved with the motion. Most of the time, we only need to know the equivalent terms to replace the linear counterpart in various equations. However, there are few finer differences that we need to be aware about. For example, how would be treat angular frequency “ω” and angular velocity of the oscillating body in SHM. They are different.

Restoring torque

We write restoring torque equation for angular SHM as :

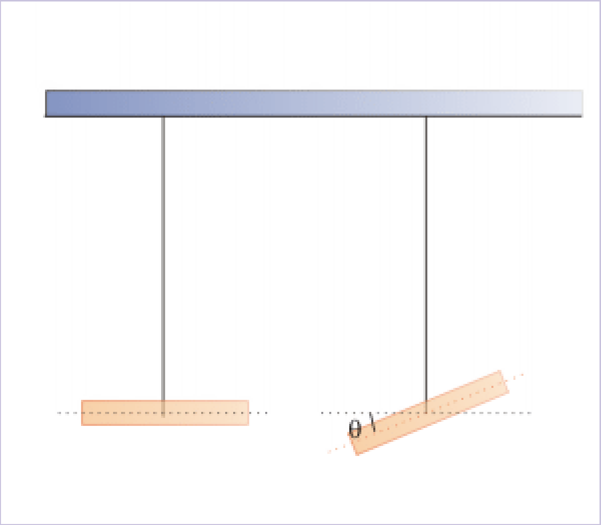
$$\tau = -k\theta$$

This “k” is springiness of the restoring torque. We associate “springiness” with any force or torque which follows the linear proportionality with negative displacement. For this reason, we call it “spring constant” for all system – not limited to block-spring system. In the case of simple pendulum, we associate this “springiness” to gravity. Similarly, this property can be associated with other forces like torsion, stress, pressure and many other force systems, which operate to restore equilibrium.

Torsion constant

A common setup capable of executing angular SHM consists of a weight attached to a wire. The rigid body is suspended from one end of the wire, whereas its other end is fixed. When the rigid body is given a small angular displacement, the body oscillates about certain reference line – which corresponds to the equilibrium position.

Torsion pendulum



The rigid body hanging from wire executes angular SHM about the position of equilibrium.

The body oscillates angularly. If we assume conservation of mechanical energy, then system oscillates with constant angular amplitude indefinitely. The whole system is known as torsion pendulum. In this case “k” of the torque equation is also known as “torsion constant”. Dropping negative sign,

$$k = \frac{\tau}{\theta}$$

Clearly, torsion constant measures the torque per radian of angular displacement. It depends on length, diameter and material of the wire.

Equations of angular SHM

We write various equations for angular SHM without derivation – unless there is differentiating aspect involved. In general, we substitute :

1. Linear inertia “m” by angular inertia “I”
2. Force, “F” by torque “τ”
3. Linear acceleration “a” by angular acceleration, “α”
4. Linear displacement “x” by angular displacement “θ”
5. Linear amplitude “A” by angular amplitude “θ_0”
6. Linear velocity “v” by angular velocity “$d\theta\ /\ dt$”

Importantly, symbols of angular frequency (ω), spring constant (k), phase constant (φ), time period (T) and frequency (ν) remain same in the description of angular SHM.

Angular displacement

$$\theta = \theta_0 \sin(\omega t + \varphi)$$

where “θ_0” is the amplitude, “φ” is the phase constant and “ωt + φ” is the phase. Clearly, angular displacement is periodic with respect to time as it is represented by bounded trigonometric function. The displacement “θ” varies between “$-\theta_0$” and “θ_0”.

SHM Equation

$$\frac{d^2\theta}{dt^2} + \omega^2\theta = 0$$

Angular velocity

Angular velocity differs to angular frequency (ω). In the case of linear SHM, we had compared linear SHM with uniform circular motion (UCM). It was found that projection of UCM on an axis is equivalent description of linear SHM. It emerged that angular frequency is same as the magnitude of constant angular velocity of the equivalent UCM.

The angular velocity of the body under angular oscillation, however, is different. Importantly, angular velocity of SHM is not constant – whereas angular frequency is constant.

The angular velocity in angular SHM is obtained either as the solution of equation of motion or by differentiating expression of angular displacement with respect to time. Clearly, we need to have a different symbol to represent angular velocity in angular SHM. Let us denote this by the differential expression itself “$\mathbb{d}\theta$ / $\mathbb{d}t$”, which is not equal to “ω”.

$$\frac{\mathbb{d}\theta}{\mathbb{d}t} = \omega\sqrt{\left(\theta_0^2 - \theta^2\right)} = \omega\theta_0\cos(\omega t + \varphi)$$

Angular acceleration

$$\alpha = -\omega^2\theta = -\frac{k}{I}\theta = -\omega^2\theta_0\sin(\omega t + \varphi)$$

Torque

$$\tau = I\alpha = -k\theta = -I\omega^2\theta == -I\omega^2\theta_0\sin(\omega t + \varphi)$$

Frequency, angular frequency, time period

The angular frequency is given by :

$$\omega = \sqrt{\left(\frac{k}{I}\right)} = \sqrt{\left|\frac{\text{angular acceleration}}{\text{angular displacement}}\right|}$$

Time period is obtained from the defining relation :

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\left(\frac{I}{k}\right)} = 2\pi\sqrt{\left|\frac{\text{angular displacement}}{\text{angular acceleration}}\right|}$$

Frequency is obtained from the defining relation :

$$\nu = \frac{1}{T} = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\left(\frac{k}{I}\right)} = \frac{1}{2\pi}\sqrt{\left|\frac{\text{angular acceleration}}{\text{angular displacement}}\right|}$$

Kinetic energy

In terms of angular displacement :

$$K = \frac{1}{2} I \left(\frac{d\theta}{dt} \right)^2 = \frac{1}{2} I\omega^2 \left(\theta_0^2 - \theta^2 \right)$$

In terms of time :

$$K = \frac{1}{2} I\omega^2 \theta_0^2 \cos^2(\omega t + \varphi)$$

Potential energy

In terms of angular displacement :

$$U = \frac{1}{2} k\theta^2 = \frac{1}{2} I\omega^2 \theta^2$$

In terms of time :

$$U = \frac{1}{2} I\omega^2 \theta_0^2 \sin^2(\omega t + \varphi)$$

Mechanical energy

$$E = K + U$$

$$E = \frac{1}{2} I\omega^2 \left(\theta_0^2 - \theta^2 \right) + \frac{1}{2} I\omega^2 \theta^2 = \frac{1}{2} I\omega^2 \theta_0^2$$

Example

Problem 1: A body executing angular SHM has angular amplitude of 0.4 radian and time period of 0.1 s. If angular displacement of the body is 0.2 radian from the center of oscillation at time t = 0, then write the equation of angular displacement.

Solution : The equation of angular displacement is given by :

$$\theta = \theta_0 \sin(\omega t + \varphi)$$

Here,

$$\theta_0 = 0.4 \quad \text{radian}$$

Also, time period is given. Hence, we can determine angular frequency, ω, as :

$$\Rightarrow \omega = \frac{2\pi}{T} = \frac{2\pi}{0.1} = 20\pi \quad \text{radian/s}$$

Putting these values, the expression of angular displacement is :

$$\Rightarrow \theta = 0.4 \sin(20\pi t + \varphi)$$

We, now, need to determine the phase constant for the given initial condition. At t = 0, θ = 0.2 radian. Hence,

$$\Rightarrow 0.2 = 0.4 \sin(20\pi X 0 + \varphi) = 0.4 \sin\varphi$$

$$\Rightarrow \sin\varphi = \frac{0.2}{0.4} = \frac{1}{2} = \sin\frac{\pi}{6}$$

$$\Rightarrow \varphi = \frac{\pi}{6}$$

Putting in the expression, the angular displacement is given by :

$$\Rightarrow \theta = 0.4 \sin\left(20\pi t + \frac{\pi}{6}\right)$$

Moment of inertia and angular SHM

Angular SHM provides a very effective technique for measuring moment of inertia. We can have a set up of torsion pendulum for a body, whose MI is to be determined. Measuring time period of oscillation, we have :

$$T = 2\pi\sqrt{\left(\frac{I}{k}\right)}$$

Squaring both sides and arranging,

$$\Rightarrow kT^2 = 4\pi^2 I$$

$$\Rightarrow I = \frac{kT^2}{4\pi^2}$$

Generally, we do not use this equation in the present form as it involves unknown spring constant, “k”. Actually, we carry out experiment for measuring time period first with a regularly shaped object like a rod of known dimensions and mass. This allows us to determine spring constant as MI of the object is known. Then, we determine time period with the object whose MI is to be determined.

Let subscripts “1” and “2” denote objects of known and unknown MIs respectively. Then,

$$\Rightarrow \frac{I_1}{I_2} = \frac{T_1^2}{T_2^2}$$

$$\Rightarrow I_2 = \frac{I_1 T_2^2}{T_1^2}$$

Example

Problem 2: A uniform rod of mass 1 kg and length 0.24 m, hangs from the ceiling with the help of a metallic wire. The time period of SHM is measured to be 2 s. The rod is replaced by a body of unknown shape and mass and the time period for the set up is measured to be 4 s. Find the MI of the body.

Solution : Let subscripts “1” and “2” denote objects of known and unknown MIs. The MI of the rod is calculated, using formula as :

$$\Rightarrow I_1 = \frac{mL^2}{12} = \frac{1X0.24^2}{12} = 0.0048 \quad kg - m^2$$

The MI of the second body is given by :

$$\Rightarrow I_2 = \frac{0.0048X4^2}{2^2} = 0.0048X4 = 0.0192 \quad kg - m^2$$

Simple and physical pendulum

Simple pendulum is an ideal oscillatory mechanism, which executes SHM. The restoring mechanism, in this case, is provided by gravitational force. Simple pendulum is simple in construction. It consists of a "particle" mass hanging from a string. The other end of the string is fixed. The overriding requirement of simple pendulum, executing SHM, can be stated in two supplementary ways :

- Mechanical energy of the oscillating system is conserved.
- The torque due to gravity (or angular acceleration) is proportional to negative of angular displacement.

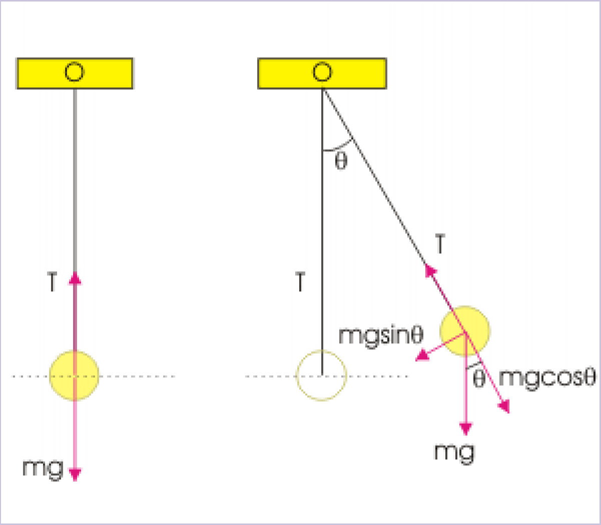
Fulfillment of the requirements, as stated, imposes certain limitations on the construction and working of simple pendulum. It can be easily inferred that we probably can not fulfill the requirement stringently, but can only approximate at the best. In this module, therefore, we shall first analyze the motion for an ideal case and then deduce conditions, which need to be fulfilled to realize ideal case to the best possible extent.

Motion of simple pendulum

There are two forces acting on the particle mass hanging from the string (called pendulum bob). One is the gravity (mg), which acts vertically downward. Other is the tension (T) in the string. In equilibrium position, the bob hangs in vertical position with zero resultant force :

$$T - mg = 0$$

Simple pendulum



Forces on the pendulum bob

At a displaced position, a net torque about the pivot point “O” acts on the pendulum bob which tends to restore its equilibrium position. In order to calculate net torque, we resolve gravity in two perpendicular components (i) mg cosθ along string and (ii) mg sinθ tangential to the path of motion.

Together with tension and two components of gravity, there are three forces acting on the pendulum bob. The line of action of tension and the component of gravity along string passes through pivot point, “O”. Therefore, torque about pivot point due to these two forces is zero. The torque on the pendulum bob is produced only by the tangential component of gravity. Hence, torque on the bob is :

$$\tau = \text{moment arm} \quad X \quad \text{Force} = -LXmg\sin\theta = -mgL\sin\theta$$

where "L" is the length of the string. We have introduced negative sign as torque is clockwise against the positive direction of displacement (anticlockwise). We can, now, use the relation “τ =Iα” to obtain the relation for angular acceleration :

$$\Rightarrow \tau = I\alpha = -mgL\sin\theta$$

$$\Rightarrow \alpha = -\frac{mgL}{I}\sin\theta$$

Clearly, this equation is not in the form “$\alpha = -\omega^2\theta$” so that pendulum bob can execute SHM. It is evident that if the requirement of SHM is to be met, then

$$\sin\theta = \theta$$

Is it possible? Not exactly, but approximately yes - if the angular displacement is a small measure. Let us check out few values using calculator :

```
--------------------------------------------
Degree Radian sine value -------------------------
------------------- 0 0.0 0.0 1 0.01746 0.017459 2
0.034921 0.034914 3 0.052381 0.052357 4 0.069841
0.069785 5 0.087302 0.087191 ---------------------
-----------------------
```

For small angle, we can consider "$\sin\theta \approx \theta$" as a good approximation. Hence,

$$\Rightarrow \alpha = -\frac{mgL}{I}\theta$$

We have just seen the condition that results from the requirement of SHM. This condition requires that angular amplitude of oscillation should be a small angle.

Angular frequency

Comparing the equation obtained for angular acceleration with that of “$\alpha = -\omega^2\theta$”, we have :

$$\Rightarrow \omega = \sqrt{\left(\frac{mgL}{I}\right)}$$

There is yet another aspect about moment of inertia that we need to discuss. Note that we have considered that bob is a point mass. In that case,

$$I = mL^2$$

and

$$\Rightarrow \omega = \sqrt{\left(\frac{mgL}{mL^2}\right)} = \sqrt{\left(\frac{g}{L}\right)}$$

We see that angular frequency is independent of mass. What happens if bob is not a point mass as in the case of real pendulum. In that case, angular frequency and other quantities dependent on angular frequency will be dependent on the MI of the bob – i.e. on shape, size, mass distribution etc.

We should understand that requirement of point mass arises due to the requirement of mass independent frequency of simple pendulum – not due to the requirement of SHM. In the nutshell, we summarize the requirement of simple pendulum that arises either due to the requirement of SHM or due to the requirement of mass independent frequency as :

- The pivot is free of any energy loss due to friction.
- The string is un-strechable and mass-less.
- There is no other force (other than gravity) due to external agency.
- The angular amplitude is small.
- The ratio of length and dimension of bob should be large so that bob is approximated as point.

Time period and frequency

Time period of simple pendulum is obtained by applying defining equation as :

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\left(\frac{L}{g}\right)}$$

Frequency of simple pendulum is obtained by apply defining equation as :

$$\nu = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\left(\frac{g}{L}\right)}$$

Special cases of simple pendulum

We have so far discussed a standard set up for the study of simple pendulum. In this section, we shall discuss certain special circumstances of simple pendulum. For example, we may be required to analyze motion of simple pendulum in accelerated frame of reference or we may be required to incorporate the effect of change in the length of simple pendulum.

Second pendulum

A simple pendulum having time period of 2 second is called “second” pendulum. It is intuitive to analyze why it is 2 second - not 1 second. In pendulum watch, the pendulum is the driver of second hand. It drives second hand once (increasing the reading by 1 second) for every swing. Since there are two swings in one cycle, the time period of second pendulum is 2 seconds.

Simple pendulum in accelerated frame

The time period of simple pendulum is affected by the acceleration of the frame of reference containing simple pendulum. We can carry out elaborate force or torque analysis in each case to determine time period of pendulum. However, we find that there is an easier way to deal with such situation. The analysis reveals that time period is governed by the “effective” acceleration or the “relative” acceleration given as :

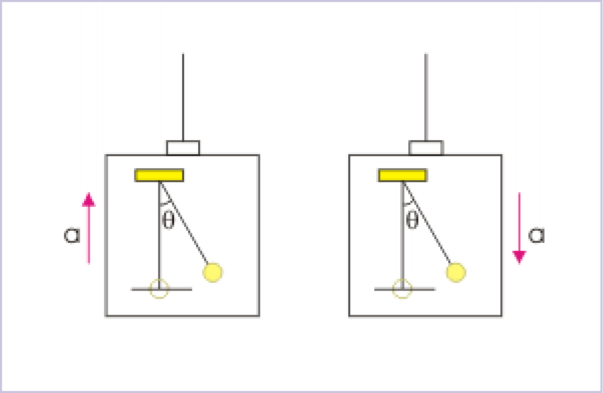
$$\mathbf{g'} = \mathbf{g} - \mathbf{a}$$

where **g’** is effective acceleration and “**a**” is acceleration of frame of reference (a≤g). We can evaluate this vector relation for different situations.

Frame of reference is accelerating upwards

Let us consider downward direction as positive. Referring left picture in the figure,

Simple pendulum in accelerated frame



The frame on left is accelerated up, whereas frame on the right is accelerated down.

$$\Rightarrow g' = g - (-a) = g + a$$

$$\Rightarrow T = 2\pi\sqrt{\left(\frac{L}{g'}\right)} = 2\pi\sqrt{\left(\frac{L}{g + a}\right)}$$

Frame of reference is accelerating downwards

Let us consider downward direction as positive. Referring right picture in the figure above,

$$\Rightarrow g' = g - a$$

$$\Rightarrow T = 2\pi\sqrt{\left(\frac{L}{g\prime}\right)} = 2\pi\sqrt{\left(\frac{L}{g-a}\right)}$$

If the simple pendulum is falling freely, then a = g,

$$\Rightarrow T = 2\pi\sqrt{\left(\frac{L}{g-g}\right)} = \text{undefined}$$

Thus, free falling pendulum will not oscillate.

Frame of reference is moving in horizontal direction

Horizontal direction is perpendicular to the vertical direction of gravity. Hence, magnitude of effective acceleration is given as :

$$\Rightarrow g\prime = \sqrt{(g^2 + a^2)}$$

$$\Rightarrow T = 2\pi\sqrt{\left(\frac{L}{g\prime}\right)} = 2\pi\sqrt{\left(\frac{L}{\sqrt{(g^2 + a^2)}}\right)}$$

Change in length

The length of simple can change due to change in temperature. The change in length is given by :

$$L\prime = L(1 + \alpha\Delta\theta)$$

Where “α” and “θ” are temperature coefficient and temperature respectively.

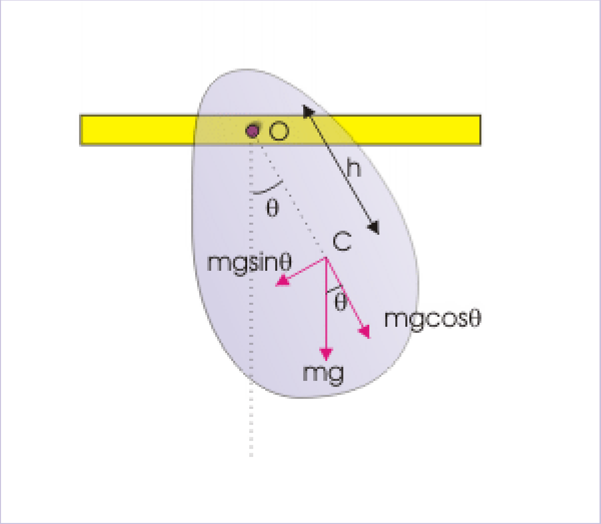
We should be careful in using this relation in case when simple pendulum is driver of a clock. For example, an increase in temperature translates into increase in length, which in turn translates into an increase in time period.

However, an increase in time period translates into loss of time on the scale of the watch. In the nutshell, watch will run slow.

Physical pendulum

In this case, a rigid body – instead of point mass - is pivoted to oscillate as shown in the figure. There is no requirement of string. As a result, there is no tension involved in this case. Besides these physical ramifications, the working of compound pendulum is essentially same as that of simple pendulum except in two important aspects :

Physical pendulum



Forces act through center of mass.

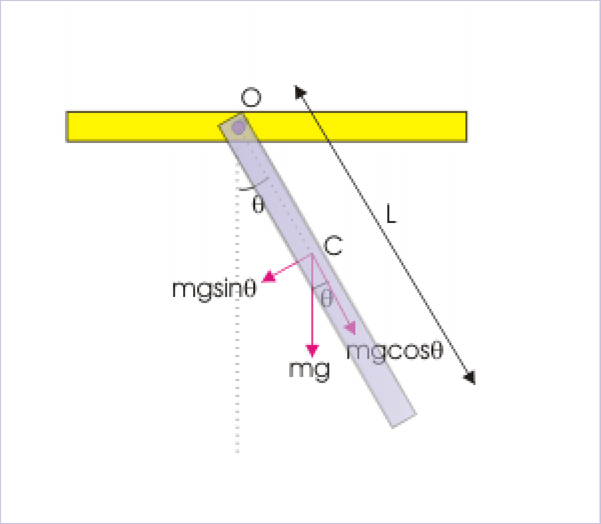
- Gravity acts through center of mass of the rigid body. Hence, length of pendulum used in equation is equal to linear distance between pivot and center of mass (“h”).
- The moment of inertia of the rigid body about point suspension is not equal to "mL^2" as in the case of simple pendulum.

The time period of compound pendulum, therefore, is given by :

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\left(\frac{I}{mgh}\right)}$$

In case we know MI of the rigid body, we can evaluate above expression of time period for the physical pendulum. For illustration, let us consider a uniform rigid rod, pivoted from a frame as shown in the figure. Clearly, center of mass is at a distance “L/2” from the point of suspension :

Physical pendulum



Forces act through center of mass.

$$h = \frac{L}{2}$$

Now, MI of the rigid rod about its center is :

$$I_C = \frac{mL^2}{12}$$

We are , however, required to evaluate MI of the rod about the point of suspension, i.e. “O”. Applying parallel axes theorem,

$$\Rightarrow I_O = \frac{mL^2}{12} + m\left(\frac{L}{2}\right)^2 = \frac{mL^2}{3}$$

Putting in the equation of time period, we have :

$$\Rightarrow T = 2\pi\sqrt{\left(\frac{I}{mgh}\right)} = 2\pi\sqrt{\left(\frac{2mL^2}{3mgL}\right)} = 2\pi\sqrt{\left(\frac{2L}{3g}\right)}$$

The important thing to note about this relation is that time period is still independent of mass of the rigid body. However, time period is not independent of mass distribution of the rigid body. A change in shape or size or change in mass distribution will change MI of the rigid body about point of suspension. This, in turn, will change time period.

Further, we should note that physical pendulum is an effective device to measure “g”. As a matter of fact, this device is used extensively in gravity surveys around the world. We only need to determine time period or frequency to determine the value of “g”. Squaring and rearranging,

$$\Rightarrow g = \frac{8\pi^2 L}{3T^2}$$

Point of oscillation

We can think of physical pendulum as if it were a simple pendulum. For this, we can consider the mass of the rigid body to be concentrated at a single point as in the case of simple pendulum such that time periods of two pendulums are same. Let this point be at a linear distance "L_o" from the point of suspension. Here,

$$\Rightarrow T = 2\pi\sqrt{\left(\frac{I}{mgh}\right)} == 2\pi\sqrt{\left(\frac{L_0}{g}\right)}$$

$$\Rightarrow Lo = \frac{Ig}{mgh}$$

The point defined by the vertical distance, "L_o", from the point of suspension is called point of oscillation of the physical pendulum. Clearly, point of oscillation will change if point of suspension is changed.

Composition of harmonic motions

Composition, here, means combining more than one simple harmonic motion. However, this statement needs to be interpreted carefully. Every particle has only one motion at a time. Actually, we mean to combine two or more harmonic motions, which result from the operation of forces, each of which is individually capable of producing SHM. Therefore, it is actually combining SHMs, produced by forces operating on a “single” particle.

Further, we need to emphasize one important limitation to our discussion. Our context is limited to combining effects of forces which produce SHMs of “same” frequency.

We organize our study under two heads :

- Composition of SHMs along same straight line.
- Composition of SHMs along two mutually perpendicular straight lines.

Force analysis

The motion of a particle, acted upon by two or more forces, is governed by Newton’s second law. The situation, here, is no different except that each of the forces is characterized for being proportional to negative of displacement. This means that each of the forces, if left to act alone, would produce SHM. Nonetheless, we have seen that force is a vector quantity with the underlying characteristic that each force produces its effect independent of other force.

The independence of a force to the presence of other forces makes our task easy to assess the net result. Let us consider the motion of a particle as if it is being worked alone by a single force. Let “ $\mathbf{r}_1$ ” and “ $\mathbf{v}_1$ ” be the position and velocity at a particular instant resulting from the action of this force. Now, let “ $\mathbf{r}_2$ ” and “ $\mathbf{v}_2$ ” be the position and velocity at a particular instant resulting from the action of the other force as if it is the only force working on the particle.

The resultant position and velocity vector, then, would simply be the vector additions of individual quantities. Hence, position and velocity of the particle, when operated simultaneously by two forces, would be :.

$$\mathbf{r} = \mathbf{r}_1 + \mathbf{r}_2$$

$$\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$$

These equations provide the general framework for studying motion that result from action of more than one force capable of producing SHM.

Composition of SHMs along same straight line

Let us consider two SHM forces, F_1 and F_2, acting along the same straight line. Let the displacements be given by two equations,

$$x_1 = A_1 \sin \omega t$$

$$x_2 = A_2 \sin(\omega t + \varphi)$$

We have written two displacements which reflect a convenient general case. Amplitudes are different. At any given instant, one of the two SHMs is "ahead of" or "lags behind" other, depending on the sign of phase constant "φ". As pointed out earlier, we have kept the angular frequency "ω" same for both SHMs.

Now, we want to find the net displacement of the particle at any given instant. Referring to our earlier discussion, we can find net displacement by evaluating vector relation,

$$\mathbf{r} = \mathbf{r}_1 + \mathbf{r}_2$$

Since both SHMs are along the same straight line, we can drop the vector sign and can simply write this relation in the present context as :

$$\Rightarrow x = x_1 + x_2$$

$$x = A_1 \sin \omega t + A_2 \sin(\omega t + \varphi)$$

Expanding trigonometric function,

$$\Rightarrow x = A_1 \sin \omega t + A_2 \sin \omega t \cos \varphi + A_2 \cos \omega t \sin \varphi$$

Segregating sine and cosine functions, keeping in mind that "φ" is constant :

$$\Rightarrow x = (A_1 + A_2 \cos \varphi) \sin \omega t + (A_2 \sin \varphi) \cos \omega t$$

The expressions in the brackets are constant. Let,

$$C = A_1 + A_2 \cos \varphi \quad \text{and} \quad D = A_2 \sin \varphi$$

Substituting in the expression of displacement, we have :

$$x = C \sin \omega t + D \cos \omega t$$

Following standard analytical method, Let

$$C = A\cos\theta \quad \text{and} \quad D = A\sin\theta$$

Substituting in the expression of displacement again, we have :

$$\Rightarrow x = A\cos\theta\sin\omega t + A\sin\theta\cos\omega t$$

$$\Rightarrow x = A\sin(\omega t + \theta)$$

This is the final expression of the composition of two SHMs in the same straight line. Clearly, the amplitude of resulting SHM is “A”. Also, the resulting SHM differs in phase with respect to either of the two SHMs. In particular, the phase of resulting SHM differs by an angle “θ” with respect of first SHM, whose displacement is given by “ $A_1 \sin\omega t$ ” We also note that frequency of the resulting SHM is same as either of two SHMs.

Phase constant

The phase constant of the resulting SHM is :

$$\Rightarrow \tan\theta = \frac{D}{C} = \frac{A_2\sin\varphi}{A_1 + A_2\cos\varphi}$$

Amplitude

The amplitude of the resultant harmonic motion is obtained solving substitutions made in the derivation.

$$C = A\cos\theta \quad \text{and} \quad D = A\sin\theta$$

$$A = \sqrt{(C^2 + D^2)} = \sqrt{\left\{(A_1 + A_2\cos\varphi)^2 + (A_2\sin\varphi)^2\right\}}$$

$$\Rightarrow A = \sqrt{\left(A^{1,2} + A_2^2\cos^2\varphi + 2A_1A_2\cos\varphi + A_2^2\sin^2\varphi\right)}$$

$$\Rightarrow A = \sqrt{\left(A_1^2 + A_2^2 + 2A_1A_2\cos\varphi\right)}$$

Important cases

We ,here, consider few interesting cases :

1: Phase difference is zero

Two SHMs are in same phase. In this case, $\cos\varphi = \cos 0^0 = 1$.

$$\Rightarrow A = \sqrt{\left(A_1^2 + A_2^2 + 2A_1A_2\cos\varphi\right)} = \sqrt{\left(A_1^2 + A_2^2 + 2A_1A_2\right)} = \sqrt{\left(A_1 + A_2\right)^2}$$

$$\Rightarrow A = A_1 + A_2$$

If additionally $A_1 = A_2$, then $A = 2A_1 = 2A_2$. Further, phase constant is given by :

$$\Rightarrow \tan\theta = \frac{D}{C} = \frac{A_2\sin\varphi}{A_1 + A_2\cos\varphi} = \frac{A_2\sin 0^0}{A_1 + A_2\cos 0^0} = 0$$

$$\Rightarrow \theta = 0$$

2: Phase difference is “π”

Two SHMs are opposite in phase. In this case, $\cos\varphi = \cos\pi = -1$.

$$\Rightarrow A = \sqrt{\left(A_1^2 + A_2^2 + 2A_1A_2\cos\varphi\right)} = \sqrt{\left(A_1^2 + A_2^2 - 2A_1A_2\right)} = \sqrt{\left(A_1 - A_2\right)^2}$$

$$\Rightarrow A = A_1 - A_2$$

The amplitude is a non-negative number. In order to reflect this aspect, we write amplitude in modulus form :

$$\Rightarrow A = |A_1 - A_2|$$

If additionally $A_1 = A_2$, then A = 0. In this case, the particle does not oscillate. Further, phase constant is given by :

$$\Rightarrow \tan\theta = \frac{D}{C} = \frac{A_2\sin\varphi}{A_1 + A_2\cos\varphi} = \frac{A_2\sin\pi}{A_1 + A_2\cos\pi} = 0$$

$$\Rightarrow \theta = 0$$

Composition by vector method

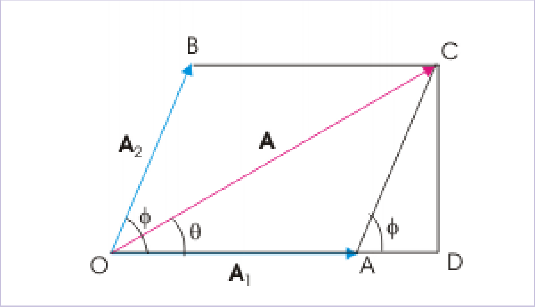
We have evaluated composition of two SHMs analytically. This has given us the detailed picture of how displacement of a particle takes place. In the nutshell, we find that resulting motion is also a SHM of same frequency as that of constituting SHMs. Besides, we are able to determine followings aspects of resulting SHM :

- Displacement
- Amplitude of the resulting SHM
- Phase constant of the resulting SHM

There is, however, an effective and more convenient alternative to determine all these aspects of SHM, using vector concept. The important thing to realize here is that amplitude can be associated with direction – apart from having magnitude. Its direction is qualified by the phase constant.

The equation of amplitude, derived earlier, provides the basis of this assumption. If we look closely at the expression of resultant amplitude, then we realize that the expression actually represents sum of two vectors namely " A_1 " and " A_2" at an angle "φ" as shown in the figure.

Composition of two SHMs



The diagonal represents the resultant amplitude.

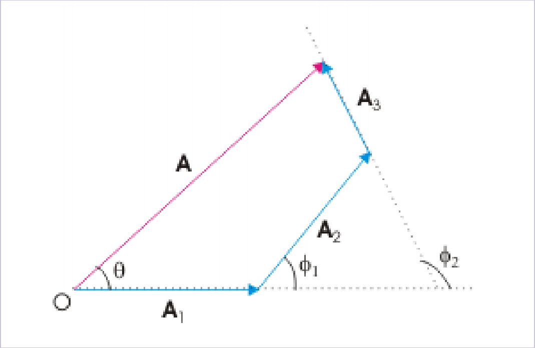
$$A = \sqrt{\left(A_1^2 + A_2^2 + 2A_1A_2\cos\varphi\right)}$$

This understanding serves our purpose. If we know amplitudes of individual SHMs and the phase difference, then we can find amplitude of the resulting SHM directly using vector sum formula. It is also evident that vector method can be used to find the resulting phase difference. From the figure,

$$\tan\theta = \frac{\text{CD}}{\text{OD}} = \frac{A_2 \sin\varphi}{A_1 + A_2\cos\varphi}$$

Vector method has one more simplifying aspect. We can compose more than two SHMs by consecutive application of parallelogram theorem or by using consecutive application of triangle law of vector addition. See the figure and observe how do we compose three SHMs and find the resulting amplitude (A) and phase difference (θ) from the reference direction.

Composition of more than two SHMs



The closing side of the ploygon represents the resultant amplitude.

Composition of SHMs in perpendicular directions

Simple harmonic motions in mutually perpendicular directions constitute a two dimensional motion. The x-direction of motion is governed by SHM force in x-direction, whereas y-direction of motion is governed by SHM force in y-direction. Let the displacements in “x” and “y” directions be :

$$x = A\sin\omega t$$

$$y = B\sin(\omega t + \varphi)$$

The force in x-direction does not affect the displacement in y-direction and vice versa. We have pointed out that “effect” of one force is independent of the presence of other forces. Here, this independence is a step ahead. Force in one direction is incapable to produce “effect” in perpendicular direction in the first place. As such, the two equations, as they are, give “x” and “y” coordinates of the particle at any given instant. The resulting motion is two dimensional motion. In the nutshell, we do not need to combine effects (displacements) in a particular direction as in the case of one dimensional composition.

Lissajous curves

The plot of the motion as defined by the SHM displacement equations is known as "Lissajous curve". In order to determine the curve (path) of resulting motion, we eliminate "t" from two equations. For this, we first expand the trigonometric expression of displacement in "y" direction :

$$y = B(\sin \omega t \cos \varphi + \cos \omega t \sin \varphi)$$

Now, we substitute the values of trigonometric functions from the expression of displacement in "x" direction. Here,

$$\sin \omega t = \frac{x}{A} \quad \text{and} \quad \cos \omega t = \sqrt{\left(1 - \frac{x^2}{A^2}\right)}$$

Substituting these values,

$$\Rightarrow y = B\left\{\frac{x}{A}\cos \varphi + \sqrt{\left(1 - \frac{x^2}{A^2}\right)}\sin \varphi\right\}$$

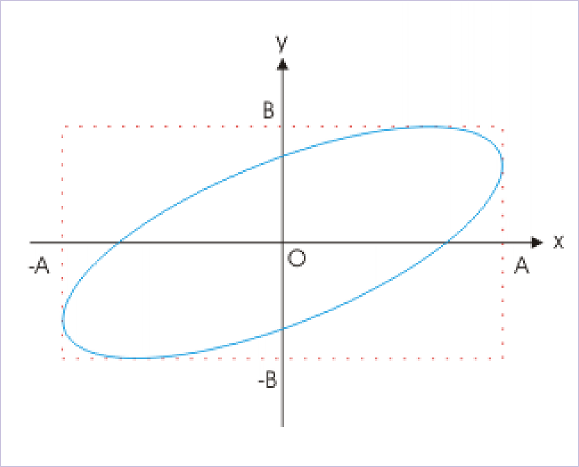
$$\Rightarrow \frac{y}{B} - \frac{x}{A}\cos \varphi = \sqrt{\left(1 - \frac{x^2}{A^2}\right)}\sin \varphi$$

Squaring both sides and rearranging,

$$\Rightarrow \frac{x^2}{A^2} + \frac{y^2}{B^2} - \frac{2xy\cos \varphi}{AB} = \sin^2 \varphi$$

This is an equation of ellipse. The nature of path depends on amplitudes of the individual SHMs and the phase difference. Importantly, we realize that resulting motion may not be oscillatory at all – although it is periodic.

Lissajous curves



The path of motion is an ellipse.

Further, we observe from the equation for displacement in “x” direction that values of “x” lie between “-A” and “A”. Similarly, values of “y” lie between “-B” and “B”. Clearly, path of resulting motion (i.e. ellipse) lies within the boundary, set up by limiting values of “x” and “y”. This is shown in the figure above.

Important cases

We ,here, consider few interesting cases :

Phase difference is zero

Zero phase difference means that individual SHMs are in phase with respect to each other. If we compare two SHMs as if they are independent and separate, then they reach mean position and respective “x” and “y” extremes at the same time. We can find the path of resulting motion by putting phase difference zero in the equation of path. Here,

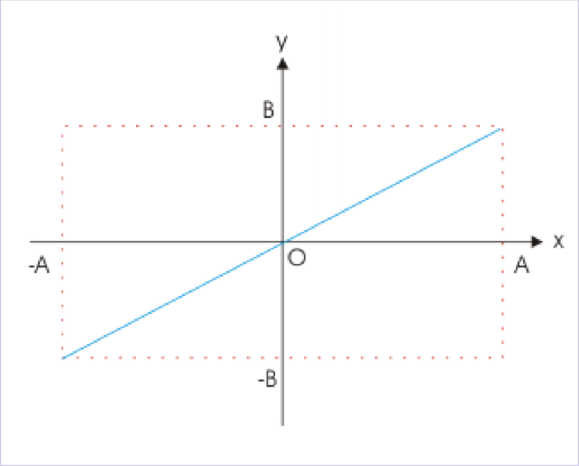
$$\Rightarrow \frac{x^2}{A^2} + \frac{y^2}{B^2} - \frac{2xy\cos 0^0}{AB} = \sin^2 0^0$$

$$\Rightarrow \frac{x^2}{A^2} + \frac{y^2}{B^2} - \frac{2xy}{AB} = 0$$

$$\Rightarrow \left(\frac{x}{A}-\frac{y}{B}\right)^2=0$$

$$\Rightarrow y=\frac{B}{A}x$$

Lissajous curves



The path of motion is a straight line.

This is the equation of a straight line. The path of motion in reference to bounding rectangle is shown for this case. It is worth pointing here that we can actually derive this equation directly simply by putting φ = 0 in displacement equations in “x” and “y” directions and then solving for "y".

Clearly, motion of the particle under this condition is an oscillatory motion along a straight line. We need to know the resultant displacement equation. Let “z” denotes displacement along the path. By geometry, we have :

$$z=\sqrt{(x^2+y^2)}$$

Substituting for “x” and “y” with φ = 0,

$$z=\sqrt{\left(A^2\sin^2\omega t+B^2\sin^2\omega t\right)}=\sqrt{(A^2+B^2)}\sin\omega t$$

This is bounded periodic harmonic sine function representing SHM of amplitude $\sqrt{(A^2+B^2)}$ and angular frequency “ω” – same as that of either of the component

SHMs.

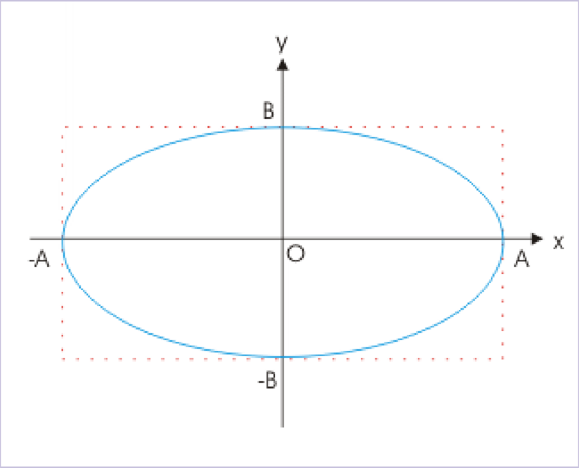
Phase difference is π/2

A finite phase difference means that individual SHMs are not in phase with respect to each other. If we compare two SHMs as if they are independent and separate, then they reach mean position and respective “x” and “y” extremes at different times. When one is at the mean position, other SHM is at the extreme position and vice versa. We can find the path of resulting motion by putting phase difference “π/2” in the equation of path. Here,

$$\Rightarrow \frac{x^2}{A^2} + \frac{y^2}{B^2} - \frac{2xy\cos\frac{\pi}{2}}{AB} = \sin^2\frac{\pi}{2}$$

$$\Rightarrow \frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$$

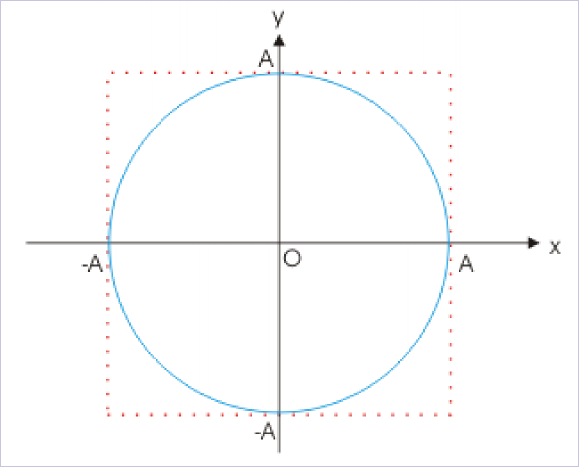
Lissajous curves



The path of motion is an ellipse.

This is an equation of ellipse having major and minor axes “2A” and “2B” respectively. The resulting motion of the particle is along an ellipse. Hence, motion is periodic, but not oscillatory. If A = B, then the equation of path reduces to that of a circle of radius “A” or “B” :

Lissajous curves



The path of motion is a circle.

$$x^2 + y^2 = A^2$$

Phase difference is π

Individual SHMs are not in phase with respect to each other. When one is at one extreme, other SHM is at the other extreme position and vice versa. We can find the path of resulting motion by putting phase difference “π” in the equation of path. Here,

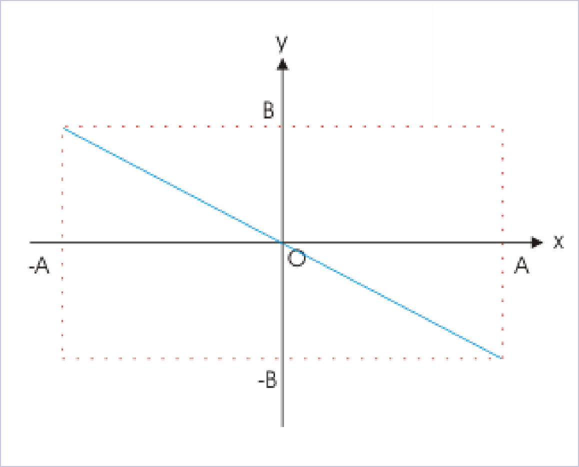
$$\Rightarrow \frac{x^2}{A^2} + \frac{y^2}{B^2} - \frac{2xy\cos\pi}{AB} = \sin^2\pi$$

$$\Rightarrow \frac{x^2}{A^2} + \frac{y^2}{B^2} + \frac{2xy}{AB} = 0$$

$$\Rightarrow \left(\frac{x}{A} + \frac{y}{B}\right)^2 = 0$$

$$\Rightarrow y = -\frac{B}{A}x$$

Lissajous curves



The path of motion is a straight line.

This is the equation of a straight line. The path of motion in reference to bounding rectangle is shown for this case. It is worth pointing here that we can actually derive this equation directly by putting "φ = π" in displacement equations in “x” and “y” directions.

Clearly, motion of the particle under this condition is an oscillatory motion along a straight line. We need to know the resultant displacement equation. Let “z” denotes displacement along the path. By geometry, we have :

$$z = \sqrt{(x^2 + y^2)}$$

$$x = A\sin\omega t$$

$$y = B\sin(\omega t + \varphi) = B\sin(\omega t + \varphi) = -B\sin\omega t$$

Substituting for “x” and “y” with φ = π,

$$\Rightarrow z = \sqrt{\left\{(A\sin\omega t)^2 + (-B\sin\omega t)^2\right\}}$$

$$\Rightarrow z = \sqrt{\left(A^2\sin^2\omega t + B^2\sin^2\omega t\right)} = \sqrt{(A^2 + B^2)}\sin\omega t$$

This is bounded periodic harmonic sine function representing SHM of amplitude $\sqrt{(A^2 + B^2)}$ and angular frequency “ω” – same as that of either of the component SHMs.

Block – spring system in SHM

Block – spring system presents an approximation of SHM. It is assumed that spring force follows Hooke’s law and there is no dissipating forces like friction or air drag. The arrangement consisting of spring with block attached at one end has many interesting variations. We analyze all such arrangements following certain basic steps to arrive at formulations for periodic attributes like time period and frequency etc.

The first step is to drive an equation between force and displacement or between acceleration and displacement. We, then, use standard expression to determine time period and frequency. Broadly, we shall be working to analyze following variations consisting of a block and spring(s) :

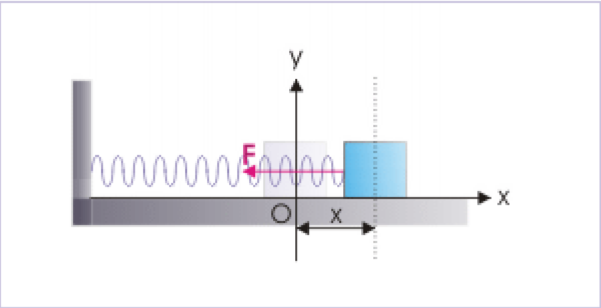
- Horizontal block – spring system
- Vertical block – spring system
- Block connected to springs in series
- Block in between two springs
- Block connected to springs in parallel

Horizontal block-spring system

The Hooke’s law governing an ideal spring relates spring force with displacement as :

$$F = -kx$$

Horizontal block-spring system



The spring is stretched a bit and then let go to oscillate.

Combining with Newton’s second law,

$$\Rightarrow F = ma = -kx$$

$$\Rightarrow a = -\frac{k}{m}x$$

Now, comparing with SHM relation “ $a = -\omega^2 x$ ”, we have :

$$\Rightarrow \omega = \sqrt{\left(\frac{k}{m}\right)} \quad \Rightarrow T = 2\pi\sqrt{\left(\frac{m}{k}\right)} \quad \Rightarrow \nu = \frac{1}{2\pi}\sqrt{\left(\frac{k}{m}\right)}$$

In these expressions “m” and “k” represent inertia and spring factor respectively.

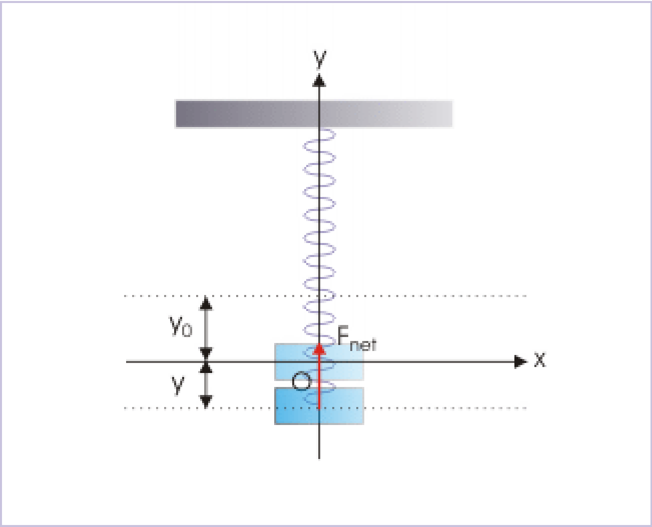
Vertical block-spring system

Vertical block-spring system differs to horizontal arrangement in the application of gravitational force. In horizontal orientation, gravitational force is perpendicular to motion and as such it is not considered for the analysis. In vertical orientation, however, the spring is in extended position due to the weight of the block before the block is set in SHM. It is in equilibrium in the extended position under the action of gravitational and spring force.

Clearly, the center of oscillation is the position of equilibrium. The block oscillates about the extended position – not about the position of neutral spring length as in the case of horizontal arrangement. Let us consider that the spring is extended by a vertical length “ y_0 ” from neutral position when it is in equilibrium position. For a further extension “y” in spring, the spring force on the block is equal to the product of spring constant and total displacement from the neutral position,

$$F = -k(y_0 + y)$$

Vertical block-spring system



The spring is stretched a bit from the equilibrium position and then let go to oscillate.

Note that we have considered downward displacement as positive. The spring force acting upward is opposite to displacement and hence negative. In this case, however, the net restoring force on the block is equal to the resultant of spring force acting upwards and gravity acting downward. Considering downward direction as positive,

$$\Rightarrow F_{\text{net}} = F + mg = -k(y_0 + y) + mg$$

But, for equilibrium position, we have the following relation,

$$mg = -ky_0$$

Substituting this relation in the expression of net restoring force, we have :

$$\Rightarrow F_{\text{net}} = -ky$$

The important point to realize here is that net restoring force is independent of gravity. It is equal to differential spring force for the additional extension – not the spring force for the total extension from the neutral position. Now, according to Newton’s second law of motion, the net restoring force is equal to the product of mass of the block and acceleration,

$$\Rightarrow F_{\text{net}} = ma = -ky$$

$$\Rightarrow a = -\frac{k}{my}$$

This relation on comparison with SHM equation “$a = -\omega^2 x$” yields same set of periodic expressions as in the case of horizontal block-spring arrangement :

$$\Rightarrow \omega = \sqrt{\left(\frac{k}{m}\right)} \quad \Rightarrow T = 2\pi\sqrt{\left(\frac{m}{k}\right)} \quad \Rightarrow \nu = \frac{1}{2\pi}\sqrt{\left(\frac{k}{m}\right)}$$

In this case, however, we can obtain alternative expressions as well for the periodic attributes as spring force at equilibrium position is equal to the weight of the block,

$$mg = -ky_0$$

Dropping negative sign and rearranging,

$$\Rightarrow \frac{m}{k} = \frac{y_0}{g}$$

Hence, the alternative expressions of periodic attributes are :

$$\Rightarrow \omega = \sqrt{\left(\frac{g}{y_0}\right)} \quad \Rightarrow T = 2\pi\sqrt{\left(\frac{y_0}{g}\right)} \quad \Rightarrow \nu = \frac{1}{2\pi}\sqrt{\left(\frac{g}{y_0}\right)}$$

Clearly, the extension of the spring owing to the weight of the block in vertical orientation has no impact on the periodic attributes of the SHM. One important difference, however, is that the center of oscillation does not correspond to the position of neutral spring configuration; rather it is shifted down by a vertical length given by :

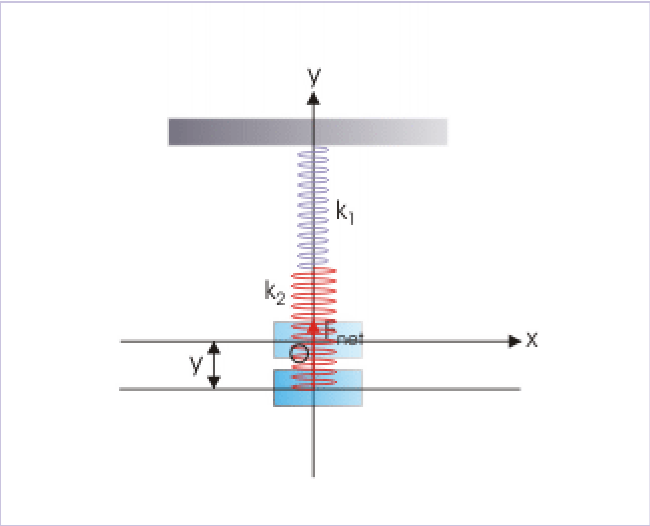
$$\Rightarrow y_0 = \frac{mg}{k}$$

Block connected to springs in series

We consider two springs of different spring constants. An external force like gravity produces elongation in both springs simultaneously. Since spring is mass-less, spring force is same everywhere in two springs. This force, however, produces different elongations in two springs as stiffness of springs are different. Let " y_1 " and " y_2" be the elongations in two springs. As discussed for the single spring, the net restoring force for each of the springs is given as :

$$F_{\text{net}} = -k_1 y_1 = -k_2 y_2$$

Block connected to springs in series



The spring is stretched a bit from the equilibrium position and then let go to oscillate.

The total displacement of the block from equilibrium position is :

$$\Rightarrow y = y_1 + y_2 = -\frac{F_{\text{net}}}{k_1} - \frac{F_{\text{net}}}{k_2}$$

$$\Rightarrow F_{\text{net}} = \frac{k_1 k_2 y}{k_1 + k_2}$$

A comparison with the expression of extension of the single spring at equilibrium position reveals that spring constant of the arrangement of two springs is equivalent to a single spring whose spring constant is given by :

$$\Rightarrow k = \frac{k_1 k_2}{k_1 + k_2}$$

This relationship can also be expressed as :

$$\Rightarrow \frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2}$$

In the nutshell, we can consider the arrangement of two springs in series as a single spring of spring constant “k”, which is related to individual spring constants by above relation. Further, we can extend this concept to a number of springs by simply extending the relation as :

$$\Rightarrow \frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} + \ldots$$

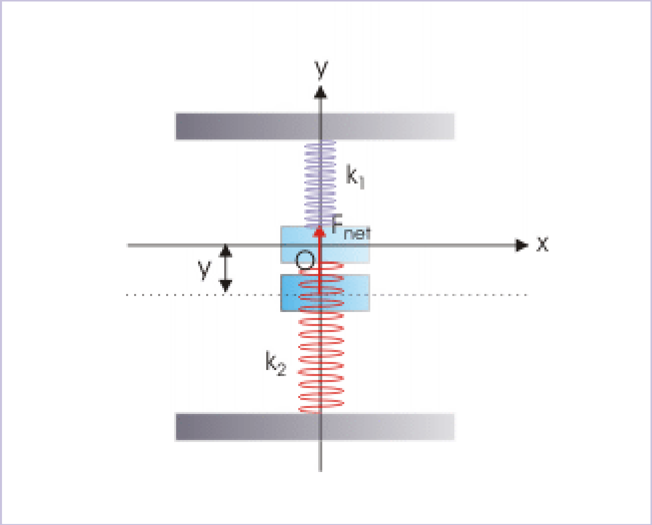
The periodic attributes are given by the same expressions, which are valid for oscillation of single spring. We only need to use equivalent spring constant in the expression.

Block in between two springs

In this arrangement, block is tied in between two springs as shown in the figure. In order to analyze oscillation, we consider oscillation from the reference position of equilibrium. Let the block is displaced slightly in downward direction (reasoning is similar if block is displaced upward). The upper spring is stretched, whereas the lower spring is compressed. The spring forces due to either of the springs act in the upward direction. The net downward displacement is related to net restoring force as :

$$F_{\text{net}} = -k_1 y_1 - k_2 y_2 = -(k_1 + k_2)y$$

Block in between two springs



The spring is stretched a bit from the equilibrium position and then let go to oscillate.

A comparison with the expression of extension of the single spring reveals that spring constant of the arrangement of two springs is equivalent to a single spring whose spring constant is given by :

$$\Rightarrow k = k_1 + k_2$$

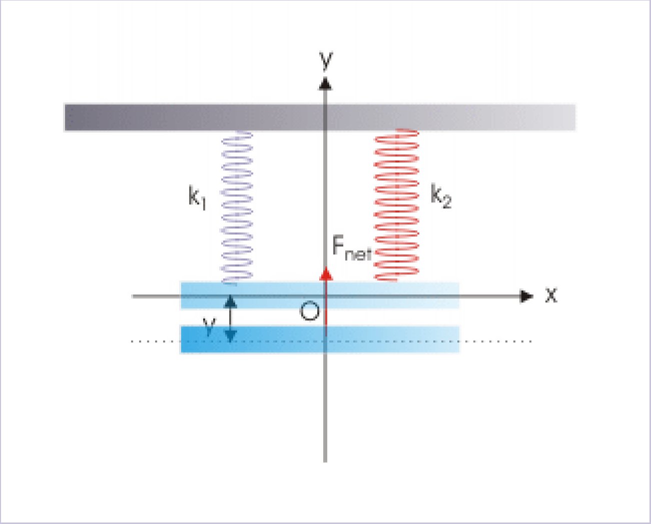
Clearly, the periodic attributes are given by the same expressions, which are valid for oscillation of single spring. We only need to use equivalent spring constant in the expression.

Block connected to springs in parallel

Here, we consider a block is suspended horizontally with the help of two parallel springs of different spring constants as shown in the figure. When the block is pulled slightly, it oscillates about the equilibrium position. The net restoring force on the block is :

$$F_{\text{net}} = -k_1 y_1 - k_2 y_2 = -(k_1 + k_2) y$$

Block connected to springs in parallel



The spring is stretched a bit from the equilibrium position and then let go to oscillate.

A comparison with the expression of extension of the single spring reveals that spring constant of the arrangement of two springs is equivalent to a single spring whose spring constant is given by :

$$\Rightarrow k = k_1 + k_2$$

Again, the periodic attributes are given by the same expressions, which are valid for oscillation of single spring. We only need to use equivalent spring constant in the expression.

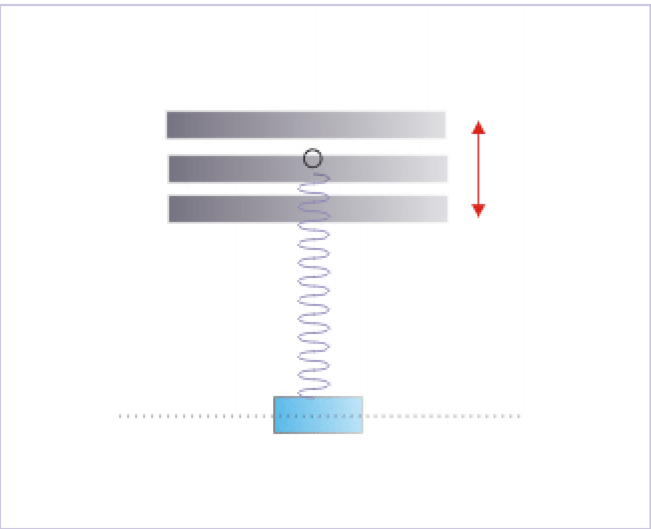
Forced oscillation

Simple harmonic motion is the simplest form of oscillation. The periodic attributes like frequency and time period are independent of disturbance that sets the periodic motion. The system oscillates at its natural frequency typical of the set up involving a restoring mechanism. Any system having moving part has inherent natural frequency or frequencies depending on the degree of freedom – numbers of possible oscillatory motions.

An oscillatory system, however, can be subjected to external force which may alter the nature of oscillation altogether. For example, a system capable of oscillation can be set to oscillate at an altogether different frequency other than natural frequency. Since we are investigating oscillatory motion, we shall study the impact of external force which is itself periodic.

Consider, for example, the oscillation of spring – block system attached to a rigid support as shown in the figure. What if the rigid support itself oscillates (up and down) at a certain frequency? What would be the motion of the block? It would be mix up of two oscillations – (i) external oscillation of the rigid support and (ii) natural oscillation of the spring – block system. In the beginning, the block will oscillate in a varying manner, but soon it settles to oscillate with frequency at which the rigid support is made to oscillate. We can describe such forced oscillation by harmonic function :

Forced oscillation



The support of spring - block system in oscillation

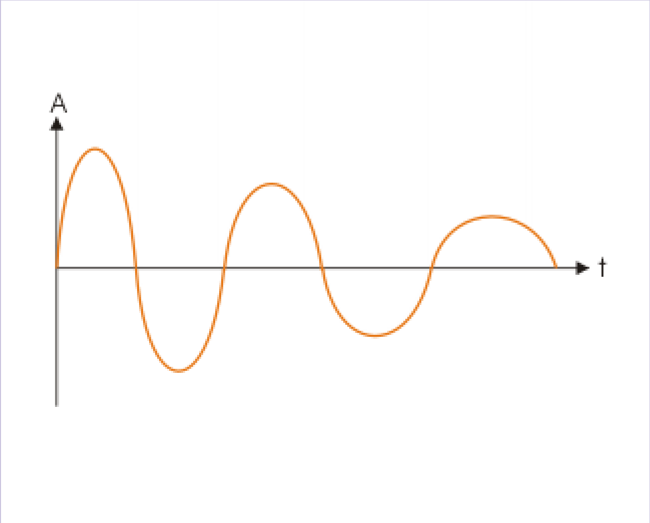
$$x = A\sin(\omega_e t + \varphi)$$

where “ ω_e” is the angular frequency of external force being applied on the system.

Damping

Damping is a condition in which external force operates in such a manner that it impedes the motion of oscillatory body. As a matter of fact, all real time harmonic motions that we consider to be simple are actually damped SHM. We consider them SHM only as an approximation. The motion of a block hanging from a spring, for example, is not SHM as air works to oppose the motion – at every instant. As a consequence, the span (amplitude) of the motion keeps decreasing every cycle. Diminishing amplitude is the characterizing feature of damped oscillation. A typical displacement – time plot looks as shown here.

Damped oscillation



The amplitude of oscillation diminishes with time.

The real time situation may be a bit complex to describe damped oscillation mathematically. Here, we consider one simplified situation in which air resistance can be considered to be proportional to the velocity of the oscillatory body. In such case, net force on the oscillating body is the resultant of restoring and damping force (with a negative sign) :

$$F_{\mathrm{net}} = -kx - bv$$

$$ma + kx + bv = 0$$

In terms of displacement derivatives :

$$\Rightarrow m\frac{\mathbb{d}^2 x}{\mathbb{d}t^2} + b\frac{\mathbb{d}x}{\mathbb{d}t} + kx = 0$$

For small damping constant “b”, the solution of this differential equation yields :

$$x = A_0 e^{-bt/2m} \sin(\omega_b t + \varphi)$$

The amplitude of the oscillation is a decreasing function in time, which tends to become zero :

$$A = A_0 e^{-\frac{bt}{2m}}$$

A damped oscillation thus dies down as the case with most of the oscillatory systems, which are not provided with external energy to compensate energy dissipated due to damping. On the other hand, the angular frequency of damped oscillation depends on additional factor of proportionality constant “b” :

$$\omega_b = \sqrt{\left\{\left(\frac{k}{m}\right) - \left(\frac{b}{2m}\right)^2\right\}}$$

Clearly, angular frequency of damped oscillation is lesser than corresponding angular frequency if damping is absent i.e. b = 0,

$$\omega_b = \sqrt{\left(\frac{k}{m}\right)} = \omega_o$$

Forced oscillation

It is clear from the discussion so far that most of artificial oscillation system tends to cease as damping is part of the natural set up. There can be various sources of damping force, but friction is one common source. There can be air resistance or resistance at the fixed hinge from which oscillating part is hung.

It is imperative that we supply appropriate energy (force) to compensate for the loss of energy due to damping. To meet this requirement, the oscillating system is subjected to oscillatory external force. The external force imparted is itself oscillatory and is, therefore, described by harmonic trigonometric function. Considering presence of damping, the force equation is :

$$F_{\text{net}} = -kx - bv + F_0 \sin \omega_e t$$

$$ma + kx + bv - F_0 \sin \omega_e t = 0$$

In terms of displacement derivatives :

$$\Rightarrow m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + b\frac{\mathrm{d}x}{\mathrm{d}t} + kx - F_0 \sin \omega_e t = 0$$

The solution of this differential equation yields :

$$x = A \sin(\omega_e t + \varphi)$$

As is evident from the expression, the system oscillates with the same frequency as that of external force. The amplitude of the oscillation is described in terms of frequency of external force (ω_e) and natural frequency (ω_o) as :

$$A = \frac{\frac{F}{m}}{\sqrt{\left\{(\omega_e^2 - \omega_o^2) + \left(\frac{b\omega_e}{m}\right)^2\right\}}}$$

We can see that this expression is a constant for given set up. It means we can sustain a constant amplitude of the oscillation by applying external oscillatory force - even if damping force is present.

Resonance

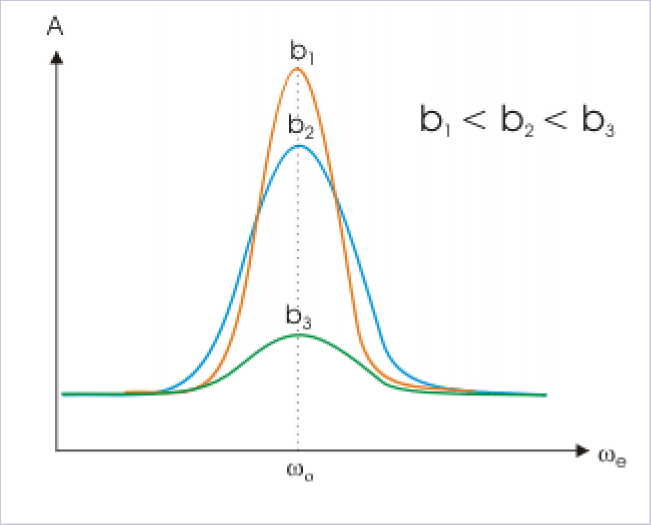
The resonance is an interesting feature of oscillation. This phenomenon attracts interest as it makes possible to achieve extra-ordinary result (material failure of large structure) with small force! Resonance also explains why earthquake causes differentiating result to different structures – most devastating where resonance occurs!

The condition for maximum amplitude is obtained by differentiating amplitude function with respect to applied frequency “ω_e” and setting the resulting expression equal to zero. This gives the resonance frequency as :

$$\omega_R = \sqrt{\left(\omega_o^2 - \frac{b^2}{4m^2}\right)}$$

If damping is absent, then the amplitude function is maximum when applied frequency is equal to natural frequency. The amplitude is infinite in such case. However, damping is always present in actuality and as such resonance amplitude is finite. For small value of damping constant, the resonance frequency is close to natural frequency and resonance is sharply defined. As the damping increases, resonance amplitude is reduced. The plots of amplitude – frequency curves outline the features of resonance amplitude as shown in the figure. Note that resonance becomes flatter as damping force increases.

Resonance



The amplitude is maximum at resonance.

Resonance underlines certain fundamental aspects of oscillatory system. First, it demonstrates that oscillation is a storing mechanism of energy. Second, it demonstrates that the energy can be supplemented in a constructive manner to increase the energy of the oscillating system to an extra-ordinary level. We can experience both these aspects easily by observing motion of swings in a nearby park. Ask a very strong adult to apply the most of his/her power to swing in one go and compare the result with that of a child who produces large swings with no such power. What is the difference? The child begins with small swing and synchronizes subsequent jerk with the oscillation. Each time the energy of the system is increased by the small amount associated with each jerk. The key, here, is the synchronization (timing of external force) and periodicity with which the jerk is applied to the swing.

Resonance is not specific to mechanical oscillatory system. The concept is equally applicable to electrical system and waves in general. Tunning of a particular radio station at a specific frequency is one of the most striking applications of this phenomenon.

We should also realize that many vibrating systems like atomic arrangements, vibrating strings etc have multiple natural frequencies. This means that these systems can be subjected to resonance at more than one frequency.

Tacoma bridge

Tacoma bridge was destroyed in 1940 just after four months of its opening. There are different theories explaining this incident. The central to these theories is resonance which could assimilate enough force from the gentle breeze and ultimately cause the material failure.

The most important design fault was the fact that bridge was reported to oscillate right from the beginning of its opening. What could not be perceived that oscillation could actually store energy in a constructive manner to such a devastating effect.

Waves

The “particle” and “wave” are two forms in which universe is manifested at the basic level. Either we interact with particle or wave. Generally, we tend to associate the concept of particle with “mass and energy” and that of “wave” with energy. In the light of Einstein formulation of “mass – energy equivalence”, we can further refine these associations :

Particle : It is aggregation or concentration of energy localized in space

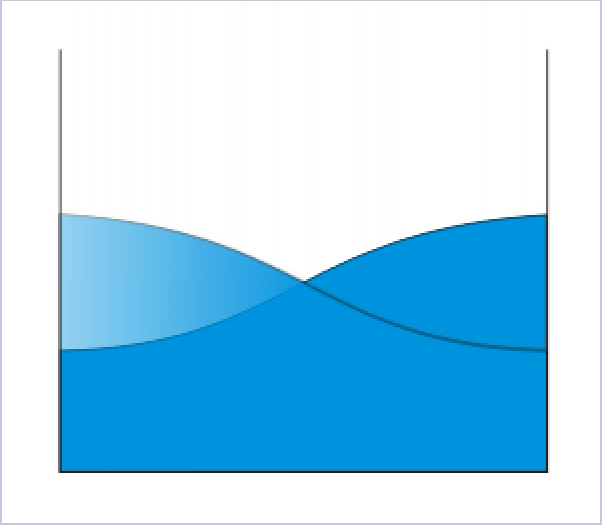
Wave : It is distribution of energy in space

What is wave?

It is a difficult question to answer. The best we can offer is interpretation which arises from explanation of various observations : Wave is distributed energy or distributed “disturbance (force)”.

Take half bucket of water. Give the bucket a jerk and watch the movement of water surface. The water mass near the surface moves from one side of the bucket to another in oscillatory fashion for some time before it comes to a standstill. The movement of water mass here replicates water waves which undergo frequent collisions with the wall of the bucket. It is easy to interpret this real time observation in terms of ideal situation. Had there been no friction (viscosity) and loss of energy during collision of water mass with the inner wall of the bucket, the disturbance (jerk) imparted would have caused a disturbance to the water body that would be sustained for all time to come.

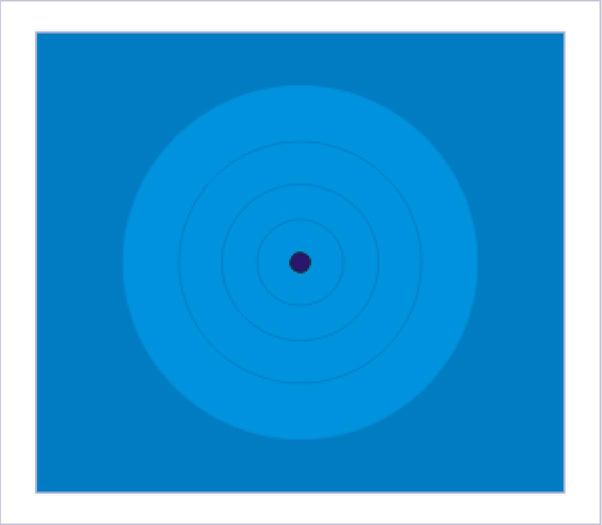
Water waves



The water mass near the surface moves from one side of the bucket to another in oscillatory fashion for some time before it comes to a standstill.

Now, we drop a pebble on a calm surface of water. The water ripples move outward in concentric circles. The concentric ripples are result of multiple disturbances created at the point where pebble hits the water surface. This is real time situation. Let us again infer the observation in terms of an ideal situation. If it is a single impact of the pebble with the surface, then a single disturbance spreads around in concentric circles and move away to cover the expanse of water surface. If it had been an infinite surface, the disturbance would have reached the farthest. We interpret this ideal situation in many different ways including :

Water waves



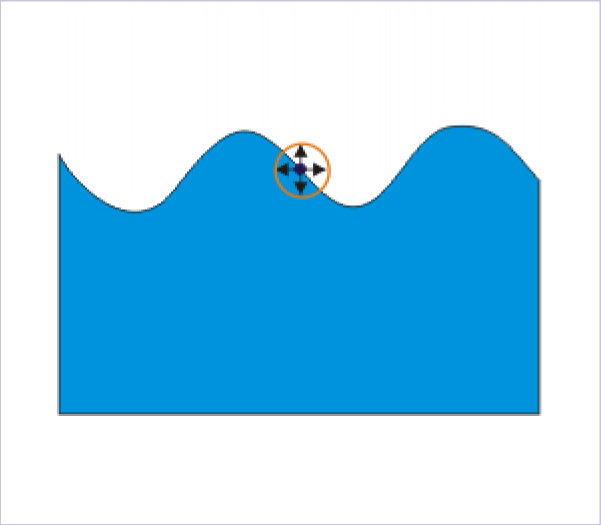
The disturbance (force) is transmitted from one point to another.

1: The disturbance (force) is transmitted from one point to another.

2: The energy is transmitted from one point to another.

If we place a small piece of paper on the water surface, then we find that paper piece moves up and down with slight side way oscillation. However, it remains where it is without any net displacement. We conclude :

3: The energy or disturbance passes in the form of wave without any net displacement of medium.
Water waves



The water particle on the surface is executing a periodic motion.

Let us now have a closer look at the motion of paper. In the ideal approximation, we see that paper is executing simple harmonic motion (SHM) in both longitudinal (side ways) and transverse (vertical) directions. The net result is that it is following an elliptical (or circular) trajectory. We infer that each water particle on the surface is executing a periodic motion. The displacement of adjacent water particles are different creating crest and trough. It appears that oscillatory motion (and hence its kinetic energy) has been passed to the next neighboring water particle and so on. Conclusion :

4: The oscillatory motion of preceding particle is imparted to the adjacent particle following it.

Returning to real time ripples on the water surface, we observe that ripples are originating continuously from the contact point for some time. It is the case of multiple disturbances. As such, a succession of ripple passes through water surface. The particle at the crest moves down to form trough subsequently and vice-versa. This cycle continues till the disturbance at the origin is over. Conclusion :

5: We need to keep creating disturbance in order to propagate wave (energy or disturbance) continuously.

Wave and medium

Water wave and propagation of sound waves show the importance of wave transmitting medium. These waves move through a medium. The Maxwell description of electromagnetic waves, however, puts an end to the requirement of a medium for the propagation of waves in general. The light wave (electromagnetic wave), for example, travels most of its journey from Sun to Earth in vacuum! The propagation of electromagnetic wave does not require a medium for propagation. On the other hand, we are unable to talk on Moon as it does not have its atmosphere to carry sound wave. Thus, we conclude that the wave can move with or without medium depending on the specific wave in question.

Disturbance

Disturbance is alteration of physical property in time. This physical quantity can be displacement, pressure, density, electric or magnetic field etc. In the propagation of water waves, we denote relative position or displacement of water molecules from the mean position as disturbance. Similar is the case with string wave – the wave traveling through a taught string. Here, displacement of string element from the undisturbed position is used to measure disturbance. We account the disturbance by air pressure in the case of sound. There is change in stress values in the case of seismic waves through Earth. On the other hand, the electromagnetic wave involves simultaneous alteration of electric and magnetic fields at right angle to each other and also to the direction of wave propagation

The most general physical parameter that can be used to represent alteration in all cases is “energy”. Instead of a particular physical parameter in specific cases, we can consider them to be alterations of “energy”. If required, we can specify the type of energy as well like : mechanical or elastic energy, pressure energy or electromagnetic energy as appropriate for a specific wave.

Dimensions of waves

The wave in a taught string moves along a straight line and as such is one-dimensional. The water wave moves over the two dimensional surface of water body and hence water wave is two-dimensional. On the other hand, sound waves or light waves propagate all around the source and hence are three dimensional in nature.

Wave classification

The waves are classified under three high level headings :

1: Mechanical waves : The motion of the particles constituting the medium follows mechanical laws i.e. Newton's laws of motion. A medium is required for propagation of mechanical waves. The motion of the waves depends on the mechanical properties of the medium.

2: Electromagnetic waves : These are propagation of electric and magnetic fields. The electromagnetic waves do not require a medium for propagation. Its speed in vacuum is a universal constant. The motion of the electromagnetic waves in a medium depends on the electromagnetic properties of the medium.

3: Matter waves : The matter wave is associated with all matter. The wave nature of matter is not always appreciable. It becomes appreciable for sub-atomic particles like electrons, which display significant wave properties. This is the basis of electron microscope.

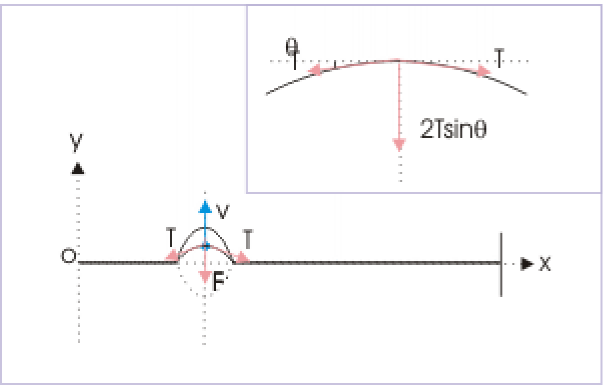
Nature of waves

The disturbance (vibration) and direction of wave may be either in the same direction (longitudinal wave) or mutually perpendicular (transverse wave). Primary seismic waves (also known as P-waves) and sound waves are examples of longitudinal waves. Electromagnetic waves are transverse waves in which electric and magnetic fields are alternating at right angles to the direction of propagation. Water waves, as discussed earlier, is combination of longitudinal and transverse waves.

Transverse waves

Transverse waves are characterized by the relatigve directions of vibration (disturbance) and wave motion. They are at right angle to each other. It is clear that vibration in perpendicular direction needs to be associated with a "restoring" mechanism in transverse direction.

Transverse waves



The vibration and wave motion are at right angle to each other (only one unit of wave shown for illustration purpose).

Consider a sinusoidal harmonic wave traveling through a string and the motion of a particle as shown in the figure above (only one unit of wave shown for illustration purpose). Since the particle is displaced from its natural (mean) position, the tension in the string arising from the deformation tends to restore the position of the particle. On the other hand, velocity of the particle (kinetic energy) moves the particle farther away from the mean position. Ultimately, the particle reaches the maximum displacement when its velocity is zero. Thereupon, the particle is pulled down due to tension towards mean position. In the process, it acquires kinetic energy (greater speed) and overshoots the mean position in the downward direction. The cycle of restoration of position continues as vibration (oscillation) of particle takes place.

For mechanical wave through solid, the restoring mechanism is provided by the elastic property of the medium. A body of liquid does not display elastic property. It can not restore deformation caused by shear i.e. transverse force. As such, transverse wave can not pass through a liquid body. On the surface of liquid, however, transverse (perpendicular to surface) deformation is restored by surface tension. For this reason, water surface waves have a transverse component. It means that transverse waves can be sustained on the surface, but not within the body of the liquid.

As far as gas is concerned, it does not exhibit elastic property like solid or surface tension like liquid. Gas simply yields to slightest transverse force without being restored. The intermolecular forces between molecules are too feeble to restore transverse deformation. Thus, mechanical transverse waves can not propagate through gas at all. Electromagnetic waves are exception to this. It is because EM waves do not require medium. There is no need to restore the system mechanically.

Longitudinal waves

Longitudinal waves are characterized by the directions of vibration (disturbance) and wave motion. They are along the same direction. It is clear that vibration in the same direction needs to be associated with a “restoring” mechanism in the longitudinal direction.

Consider the case of sound wave. The wave comprises alternating compressions and rarifications. The compressed zone is characterized by higher pressure, which tends to expand the air in the zone. Thus, there is alteration of pressure as the zone transitions from compression to rarification and so on.

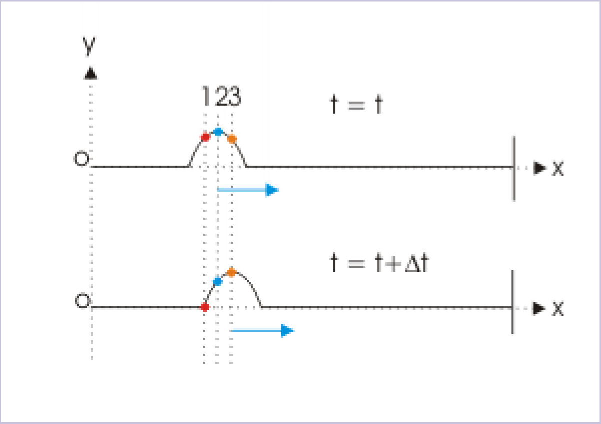
It is intuitive to note that all medium types (solid, liquid or gas) support longitudinal waves.

Mathematical description of waves

We shall attempt here to evolve a mathematical model of a traveling transverse wave. For this, we choose a specific set up of string and associated transverse waves traveling through it. The string is tied to a fixed end, while disturbance is imparted at the free end by up and down motion. For our purpose, we consider that pulse is small in dimension; the string is light, elastic and homogeneous. These assumptions are required as we visualize a small traveling pulse which remains undiminished when it moves through the string. We also assume that the string is long enough so that our observation is not subject to pulse reflected at the fixed end.

For understanding purpose, we first consider a single pulse as shown in the figure (irrespective of whether we can realize such pulse in practice or not). Our objective here is to determine the nature of a mathematical description which will enable us to determine displacement (disturbance) of string as pulse passes through it. We visualize two snap shots of the traveling pulse at two close time instants “t” and “t+Δt”. The single pulse is moving towards right in the positive x-direction.

Transverse waves



The vibration and wave motion

are at right angle to each other.

Three positions along x-axis named “1”,”2” and “3” are marked with three vertical dotted lines. At either of two instants as shown, the positions of string particles have different displacements from the undisturbed position on horizontal x-axis. We can conclude from this observation that displacement in y-direction is a function of positions of particle in x-direction. As such, the displacement of a particle constituting the string is a function of “x”.

Let us now observe the positions of a given particle, say “1”. It has certain positive displacement at time t = t. At the next snapshot at t=t+Δt, the displacement has reduced to zero. The particle at “2” has maximum displacement at t=t, but the same has reduced at t=t+Δt. The third particle at “3” has certain positive displacement at t=t. At t=t+Δt, it acquires additional positive displacement and reaches the position of maximum displacement. From these observations, we conclude that displacement of a particle at any position along the string is a function of “t”.

Combining two observations, we conclude that displacement of a particle is a function of both position of the particle along the string and time.

$$y = y(x, t)$$

We can further specify the nature of the mathematical function by associating the speed of the wave in our consideration. Let “v” be the constant speed with which wave travels from the left end to the right end. We notice that wave function at a given position of the string is a function of time only as we are considering displacement at a particular value of “x”. Let us consider left hand end of the string as the origin of reference (x=0 and t=0). The displacement in y-direction (disturbance) at x=0 is a function of time, “t” only :

$$y = f(t)$$

The disturbance travels to the right at a constant speed “v”. Let it reaches a point specified as x=x after time “t”. If we visualize to describe the origin of this disturbance at x=0, then time elapsed for the disturbance to move from the origin (x=0) to the point (x=x) is “x/v”. Therefore, if we want to use the function of displacement at x=0 as given above, then we need to subtract the time elapsed and set the equation as :

$$y = f\left(t - \frac{x}{v}\right) = f\left\{\frac{(vt - x)}{v}\right\}$$

Division by a constant like speed of wave does not change the nature of argument. Hence :

$$y = f(vt - x)$$

We can exchange terms of the argument of the wave function as well :

$$y = f(x - vt)$$

The exchange of terms aound negative sign introduces a phase difference between waves represented by two forms. The coorect order of terms depend on the state of motion of the particle at x=0 and t=0. We shall explore this aspect in detail when cosidering transverse harmonic waves in a separate module.

The wave functions represent displacement of unidirectional wave. This describes the displacement (disturbance) along the string which moves in positive x-direction. Extending the derivation, the function representing a wave moving in negative x-direction is :

$$y = f(x + vt)$$

It is the sign (minus or plus) separating two terms of the argument that determines the direction of wave with respect to positive x-direction. We should, however, be careful that all function of the forms as indicated above may not represent a wave. For example, the function should evaluate to a finite value as displacement needs to be finite. For this reason, a function given here under is invalid wave function (not defined for x=0 and t =0) :

$$y = \frac{1}{x + vt}$$

On the other hand, following functions are bounded by finite values and hence are valid wave function :

$$y = A_o e^{-\left(t - \frac{x}{\lambda/T}\right)}$$

$$y = A_o \sin(\omega t - kx)$$

Wave equation

The propagation of wave is governed by a differential equation. For wave in one dimension, the equation is given as :

$$\frac{\mathbb{d}^2 u}{\mathbb{d}t^2} = k \frac{\mathbb{d}^2 u}{\mathbb{d}x^2}$$

Here “k” is a constant equal to square of the wave speed. The parameter “u” is disturbance or amplitude parameter, which can be displacement, pressure or electric field depending on the wave in question. The parameter “u” is a function of position (x) and time (t). This equation can be written with respect to “y” and “z” directions to represent two or three dimensional waves.

Confining ourselves to one dimensional wave, putting $k = v^2$ and u = y for displacement from the mean position, the differential equation takes the form :

$$\Rightarrow \frac{\mathbb{d}^2 y}{\mathbb{d}t^2} = v^2 \frac{\mathbb{d}^2 y}{\mathbb{d}x^2}$$

A general solution of the wave equation in one dimension is given as :

$$y(x, t) = f(ax \pm bt)$$

The solution for the wave moving in positive x-direction is :

$$y(x,t) = f(ax - bt)$$

A comparison of this function with the equation derived earlier $y = f(x - vt)$, we see that speed of the wave is :

$$\Rightarrow v = \frac{dx}{dt} = \frac{b}{a}$$

Example:

Problem : Find the speed of the wave which is represented by the function :

$$y = \frac{1}{(2x + 3t)^2 + 2}$$

where “x” is in meters and “t” is in second.

Solution : The wave function is finite for all values of “t” and “x”. It has the form y= f(ax+bt). It means that the wave is traveling opposite to the positive x-direction. The speed of the wave is :

$$v = \frac{b}{a} = \frac{3}{2} = 1.5 \quad m/s$$

Interpreting wave function

What does wave function represent? Here, it is helpful to recall that wave, after all, is energy. A wave function should represent the wave (energy form) – not motion of a particle. We shall see, here, that wave function - apart from describing motion of individual particles - also represents the motion of “disturbance” as required. It represents something which has no material existence. How does it do so?

In the earlier section, the mathematical wave function has been developed to determine disturbance at any position “x” and time instant “t”. Note the important point – it gives disturbance of all particles on the string at any given instant – not a single particle. In order to understand the difference, let us consider a sinusoidal function valid for wave on a taught string :

$$y(x,t) = A\sin k(x - vt)$$

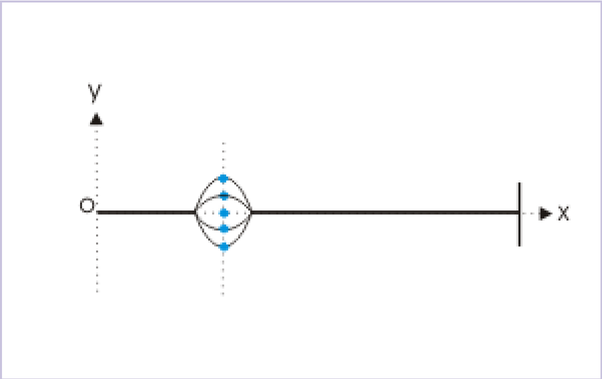
At the point of the origin, x = 0, the function is :

$$y(x = 0,t = t) = A\sin(-kvt) = -A\sin kvt$$

This function is a sine function in time “t”. As we are aware, this is an equation of simple harmonic motion. It describes the motion of the particle executing SHM at x=0. Similarly, the function representing motion of the particle in “y” direction at x = a is :

$$y(x = a, t = t) = A\sin k(a - vt)$$

Motion of a particle



The particle executes SHM (only one unit of wave shown for illustration purpose).

Clearly, the sinusoidal expression at a given position describes motion of a single particle at that point – not that of wave i.e. the motion of disturbance.

The speed of the particle in y-direction is given by the time derivative of particle's displacement from its mean position:

$$v_p = \frac{dy}{dt}$$

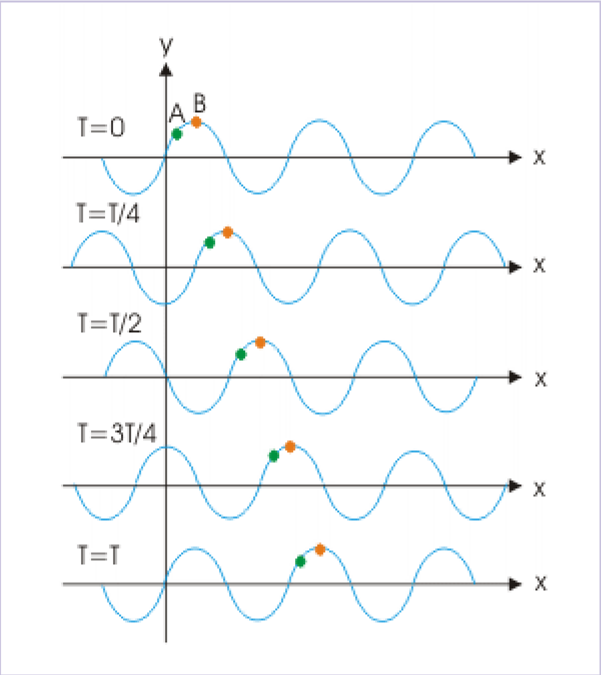
Thus, we see that we can use wave function to interpret the motion associated with the particle or small string segment in y-direction.

Now, let us look at the function as a function of "x" along which wave is considered to travel. At time instant t = 0, the shape of the wave is sinusoidal and the equation representing the shape is :

$$y(x = x, t = 0) = A\sin kx$$

The snapshots corresponding to time instants t = 0,T/4, T/2, 3T/4 and T, where "T" is the time period of oscillation in y –direction, are shown in the figure. Looking at the figures, let us ask this question to ourselves: what does change with "x"? We know that string particle is not moving in x-direction. Definitely, an expression involving change in "x" does not represent motion of a particle.

Wave motion



The snapshots at different time instants.

Looking closely at the figure, we note that a disturbance shown by letter “A” has moved in the positive x-direction at the successive time instants. We can make the same observation with respect to the maximum displacement at “B”. These markings at “A” and “B” here show the amount of disturbance i.e. displacement in y- direction. They have actually moved to right in the successive snapshots. Key here is to understand that we are talking about disturbance represented by letters “A” and “B” – not the particles at those points. We should keep in our mind that string particle has no lateral movement in x-direction. Keeping this in our mind, we can say that motion in x-direction, as described by the wave function, represents motion of a disturbance or more sophistically the motion of a wave. As such, the speed of wave is given by :

$$v = \frac{dx}{dt}$$

This expression gives the speed of the “disturbance” or plainly speaking “wave”. In other words, mathematical representation of wave (energy) is equivalent to the motion of "disturbance". This concept of wave motion is further verified by emphasizing that the argument of sinusoidal function is constant for a given “disturbance” as it travels in x-direction (Note that we are observing motion of a particular disturbance). This implies :

$$y(x,t) = A \sin k(x - vt) = \text{constant}$$

$$k(x - vt) = \text{constant}$$

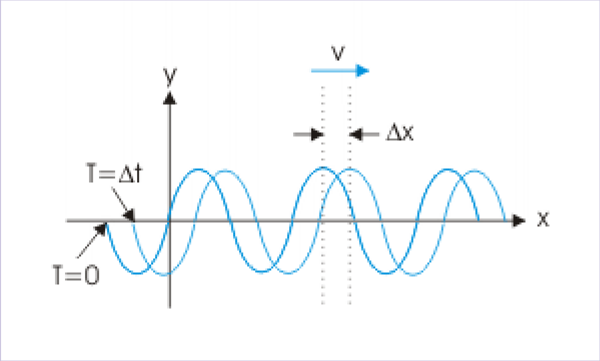
Differentiating with respect to “x”,

$$k\frac{dx}{dt} = kv$$

$$v = \frac{dx}{dt}$$

We can look at wave motion in yet another way. It can be considered to be the motion of the “wave form” or “the shape of the disturbance”. Look at the figure below. A wave form at an instant is displaced by a distance “Δx” in time interval “Δt”. The speed, at which this wave form is moving, is again obtained by the ratio of displacement in x-direction and time interval or as the time derivative of “x”.

Wave motion



The snapshots at different time instants.

Transverse harmonic waves

In this module, we shall consider wave motion resulting from harmonic vibrations. We shall discuss harmonic transverse wave in the context of a string. We assume that there is no loss of energy during transmission of wave along the string. This can be approximated when the string is light and taught. In such condition if we oscillate the free end in harmonic manner, then the vibrations in the string are simple harmonic motion perpendicular to the direction of wave motion. The amplitude of wave form remains intact through its passage along the string

Harmonic wave function

We know that a wave function representing motion in x-direction has the form :

$$y(x,t) = Af(ax - bt)$$

For the case of harmonic vibration, we represent harmonic wave motion in terms of either harmonic sine or cosine function :

$$y(x,t) = A\sin(kx - \omega t)$$

The argument of trigonometric function “kx-ωt” denotes the phase of the wave function. The phase identifies displacement (disturbance) at a particular position and time. For a particular displacement in y-direction, this quantity is a constant :

$$kx - \omega t = \text{constant}$$

where “k” is known as wave number. Its unit is radian/ meter so that the terms in the expression are consistent. Importantly, “k” here is not spring constant as used in the description of SHM.

Amplitude

The amplitude is the maximum displacement on either side of mean position and is a positive quantity. Since sine or cosine trigonometric functions are bounded between “-1” and “1”, the y-displacement is bounded between “-A” and “A”.

Harmonic oscillatory properties

Each particle (or a small segment of string) vibrates in simple harmonic motion. The particle attains the greatest speed at the mean position and reduces to zero at extreme positions. On the other hand, acceleration of the particle is greatest at extreme positions and zero at the mean position. The vibration of particle is represented by a harmonic sine or cosine function. For x=0 :

$$y(x=0,t=t)=A\sin(-\omega t)=-A\sin\omega t$$

Clearly, the displacement in y-direction is described by the bounded sine or cosine function. The important point here is to realize that oscillatory attributes (like time period, angular and linear frequency) of wave motion is same as that of vibration of a particle in transverse direction. We know that time period in SHM is equal to time taken by the particle to complete one oscillation. It means that displacement of the particle from the mean position at a given position such as x=0 has same value after time period “T” :

$$y(x=0,t=t)=y(x=0,t=t+T)$$

$$\Rightarrow -A\sin\omega t=-A\sin\omega(t+T)=-A\sin(\omega t+\omega T)$$

We know that sine of an angle repeats after every “2π” angle. Therefore, the equality given above is valid when :

$$\Rightarrow \omega T=2\pi$$

$$\Rightarrow \omega=\frac{2\pi}{T}$$

The linear frequency is given by :

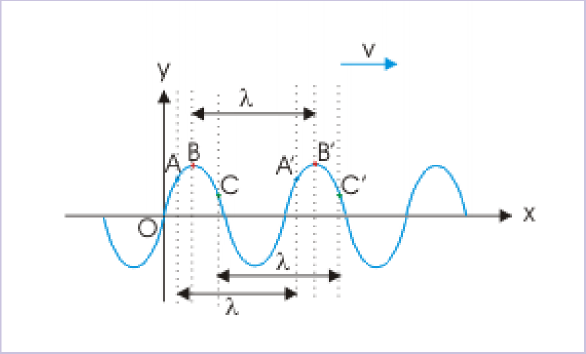
$$\Rightarrow \nu = \frac{1}{T} = \frac{\omega}{2\pi}$$

Wavelength and wave number

Wavelength is unique to the study of wave unlike frequency and time period, which is common to oscillatory motion of a particle of the string.

The wavelength is equal to linear distance between repetitions of transverse disturbance or phase. In terms of the speed of the wave, the wavelength of a wave is defined as the linear distance traveled by the wave in a period during which one cycle of transverse vibration is completed. We need to emphasize here that we can determine this wave length by choosing any point on the wave form. Particularly, it need not be a crest or trough which is otherwise a convenient reference point for measuring wavelength. For example, we can consider points A, B and C as shown in the figure. After a time period, T, points marked A’, B’ and C’ are at a linear distance of a wavelength, denoted by “λ”. Clearly, either of linear distances AA’ or BB’ or CC’ represents the wavelength.

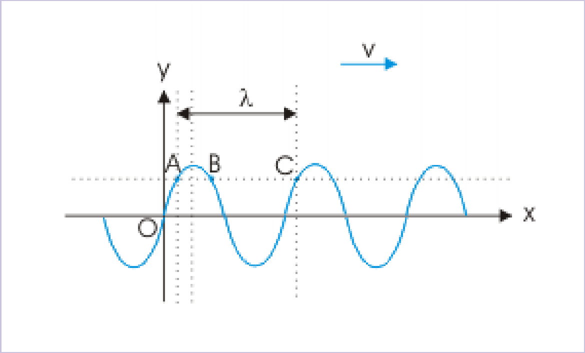
Wavelength



The wavelength is equal to linear distance between repetitions of transverse disturbance or phase.

It should be clearly understood that repetition of phase (displacement) is not identified by the magnitude of displacement alone. In the figure shown below, the points “A”, “B” and ”C” have same y-displacements. However, the phase of the waveform at “B” is not same as that at “A”. Note that sense of vibration at “A” and “B” are different (As a matter of fact, the particle at "A" is moving down whereas particle at "B" is moving upward. We shall discuss sense of vibrations at these points later in the module). Since phase is identified by the y-displacement and sense of vibration at a particular position, we need to compare the sense along with the magnitude of y-displacement as well to find the wavelength on a waveform. The wavelength, therefore, is equal to linear distance between positions of points “A” and “C”.

Wavelength



The wavelength is equal to linear distance between repetitions of transverse disturbance or phase.

$$AC = \lambda$$

Mathematically, let us consider y-displacement at a position specified by x=x. At time t = 0, the y-displacement is given by :

$$y(x = x, t = 0) = A \sin kx$$

According to definition of wavelength, the y-displacement is same at a further distance “λ”. It means that :

$$\Rightarrow y(x = x, t = 0) = y(x = x + \lambda, t = 0)$$

$$\Rightarrow A\sin kx = A\sin k(x+\lambda) = A\sin(kx+k\lambda)$$

The equality given above is valid when :

$$\Rightarrow k\lambda = 2\pi$$

$$\Rightarrow k = \frac{2\pi}{\lambda}$$

Speed of transverse harmonic wave

We can determine speed of the wave by noting that wave travels a linear distance "λ" in one period (T). Thus, speed of wave is given by :

$$v = \frac{\lambda}{T} = \nu\lambda = \frac{\frac{2\pi}{k}}{\frac{2\pi}{\omega}} = \frac{\omega}{k}$$

It is intuitive to use the nature of phase to determine wave speed. We know that phase corresponding to a particular disturbance is a constant :

$$kx - \omega t = \text{constant}$$

Speed of wave is equal to time rate at which disturbance move. Hence, differentiating with respect to time,

$$\Rightarrow k\frac{dx}{dt} - \omega = 0$$

Rearranging, we have :

$$\Rightarrow v = \frac{dx}{dt} = \frac{\omega}{k}$$

Here, "ω" is a SHM attribute and "k" is a wave attribute. Clearly, wave speed is determined by combination of SHM and wave attributes.

Example:
Problem : The equation of harmonic wave is given as :

$$y(x,t) = 0.02\sin\left\{\frac{\pi}{6}(8x - 24t)\right\}$$

All units are SI units. Determine amplitude, frequency, wavelength and wave speed
Solution : Simplifying given equation :

$$y(x,t) = 0.02\sin\left\{\frac{\pi}{3}(12x - 24t)\right\} = 0.02\sin\left\{(4\pi x - 8\pi t)\right\}$$

Comparing given equation with standard equation :

$$y(x,t) = A\sin(kx - \omega t)$$

Clearly, we have :

$$\Rightarrow A = 0.02 \quad m$$

$$\Rightarrow \omega = 2\pi\nu = 8\pi$$

$$\Rightarrow \nu = 4 \quad Hz$$

Also,

$$\Rightarrow k = \frac{2\pi}{\lambda} = 4\pi$$

$$\Rightarrow \lambda = 0.5 \quad m$$

Wave speed is given as :

$$\Rightarrow v = \frac{\omega}{k} = \frac{8\pi}{0.5} = 16\pi \quad \frac{m}{s}$$

Initial phase

At x=0 and t =0, the sine function evaluates to zero and as such y-displacement is zero. However, a waveform can be such that y-displacement is not zero at x=0 and t=0. In such case, we need to account for the displacement by introducing an angle like :

$$y(x,t) = A\sin(kx - \omega t + \varphi)$$

where "φ" is initial phase. At x=0 and t=0,

$$y(0{,}0) = A\sin(\varphi)$$

The measurement of angle determines following two aspects of wave form at x=0, t=0 : (i) whether the displacement is positive or negative and (ii) whether wave form has positive or negative slope.

For a harmonic wave represented by sine function, there are two values of initial phase angle for which displacement at reference origin (x=0,t=0) is positive and has equal magnitude. We know that the sine values of angles in first and second quadrants are positive. A pair of initial phase angles, say φ = π/3 and 2π/3, correspond to equal positive sine values as :

$$\sin\theta = \sin(\pi - \theta)$$

$$\sin\frac{\pi}{3} = \sin\left(\pi - \frac{\pi}{3}\right) = \sin\left(\frac{2\pi}{3}\right) = \frac{1}{2}$$

We can determine the slope of the waveform by partially differentiating y-function with respect to "x" (considering "t" constant) :

$$\Rightarrow \frac{\partial y}{\partial t} = \frac{\partial}{\partial t}A\sin(kx - \omega t + \varphi) = kA\cos(kx - wt + \varphi)$$

$$\text{At} \quad x = 0{,}t = 0{,}\varphi = \pi/3$$

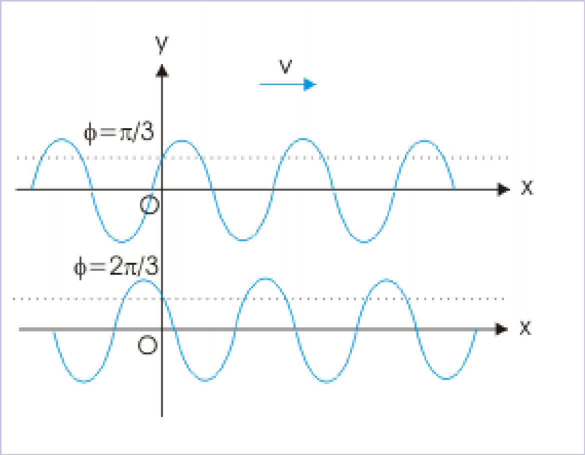
$$\Rightarrow \text{Slope} = kA\cos\varphi = kA\cos\frac{\pi}{3} = kAX\frac{\sqrt{3}}{2} = \text{a positive number}$$

$$\text{At} \quad x = 0{,}t = 0{,}\varphi = 2\pi/3$$

$$\Rightarrow \text{Slope} = kA\cos\varphi = kA\cos\frac{2\pi}{3} = -kAX\frac{\sqrt{3}}{2} = \text{a negative number}$$

The smaller of the two (π/3) in first quadrant indicates that wave form has positive slope and is increasing as we move along x-axis. The greater of the two (2π/3) similarly indicates that wave form has negative slope and is decreasing as we move along x-axis. The two initial wave forms corresponding to two initial phase angles in first and second quadrants are shown in the figure below.

Initial phase



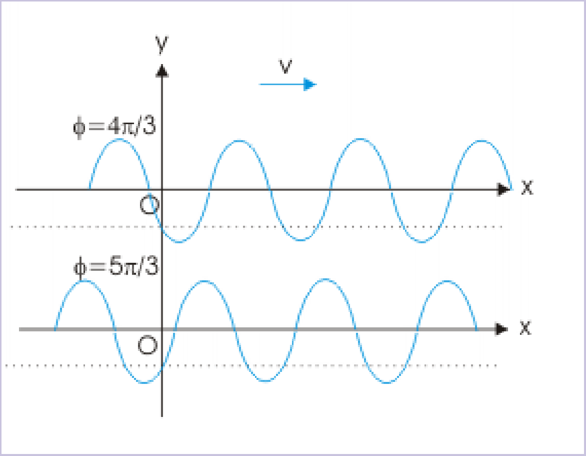
Initial phase angles in first and second quadrant.

We can interpret initial phase angle in the third and fourth quadrants in the same fashion. The sine values of angles in third and fourth quadrants are negative. There is a pair of two angles for which sine has equal negative values. The angle in third quarter like 4π/3 indicates that wave form has negative slope and is further decreasing (more negative) as we move along x-axis. On the other hand, corresponding angle in fourth quadrant for which magnitude is same is 5π/3.

$$\sin\frac{4\pi}{3} = \sin\frac{5\pi}{3} = -\frac{1}{2}$$

For initial phase angle of 5π/3 in fourth quadrant, the wave form has positive slope and is increasing (less negative) as we move along x-axis. The two initial wave forms corresponding to two initial phase angles in third and fourth quadrants are shown in the figure below.

Initial phase



Initial phase angles in third and fourth quadrant.

We can also denote initial phase angles in third and fourth quadrants (angles greater than "π") as negative angles, measured clockwise from the reference direction. The equivalent negative angles for the example here are :

$$-\left(2\pi-\frac{4\pi}{3}\right)=\frac{4\pi}{3}-2\pi=-\frac{2\pi}{3}$$

and

$$-\left(2\pi-\frac{5\pi}{3}\right)=\frac{5\pi}{3}-2\pi=-\frac{\pi}{3}$$

Particle velocity and acceleration

Particle velocity at a given position x=x is obtained by differentiating wave function with respect to time "t". We need to differentiate equation by treating “x” as constant. The partial differentiation yields particle velocity as :

$$v_p = \frac{\partial}{\partial t} y(x,t) = \frac{\partial}{\partial t} A\sin(kx - \omega t) = -\omega A\cos(kx - \omega t)$$

We can use the property of cosine function to find the maximum velocity. We obtain maximum speed when cosine function evaluates to “-1” :

$$\Rightarrow v_{p\mathrm{max}} = \omega A$$

The acceleration of the particle is obtained by differentiating expression of velocity partially with respect to time :

$$\Rightarrow a_p = \frac{\partial}{\partial t} v_p = \frac{\partial}{\partial t}\left\{-\omega A\cos(kx - \omega t)\right\} = -\omega^2 A\sin(kx - \omega t) = -\omega^2 y$$

Again the maximum value of the acceleration can be obtained using property of sine function :

$$\Rightarrow a_{p\mathrm{max}} = \omega^2 A$$

Relation between particle velocity and wave (phase) speed

We have seen that particle velocity at position “x” and time “t” is obtained by differentiating wave equation with respect to “t”, while keeping time “x” constant :

$$v_p = \frac{\partial}{\partial t} y(x,t) = -\omega A\cos(kx - \omega t)$$

On the other hand, differentiating wave equation with respect to “x”, while keeping “t”, we have :

$$\frac{\partial}{\partial x} y(x,t) = kA\cos(kx - \omega t)$$

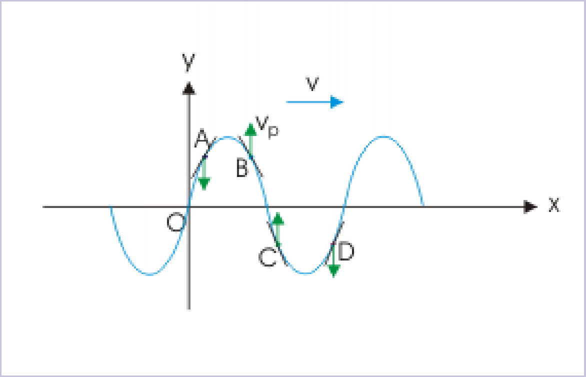
The partial differentiation gives the slope of wave form. Knowing that speed of the wave is equal to ratio of angular frequency and wave number, we divide first equation by second :

$$\Rightarrow \frac{v_p}{\frac{\partial y}{\partial x}} = -\frac{\omega}{k} = -v$$

$$\Rightarrow v_p = -v\frac{\partial y}{\partial x}$$

At a given position “x” and time “t”, the particle velocity is related to wave speed by this equation. Note that direction of particle velocity is determined by the sign of slope as wave speed is a positive quantity. We can interpret direction of motion of the particles on the string by observing “y-x” plot of a wave form. We know that “y-x” plot is a description of wave form at a particular time instant. It is important to emphasize that a wave like representation does not show the motion of wave. An arrow showing the direction of wave motion gives the sense of motion. The wave form is a snapshot (hence stationary) at a particular instant. We can, however, assess the direction of particle velocity by just assessing the slope at any position x=x. See the plot shown in the figure below :

Particle velocity

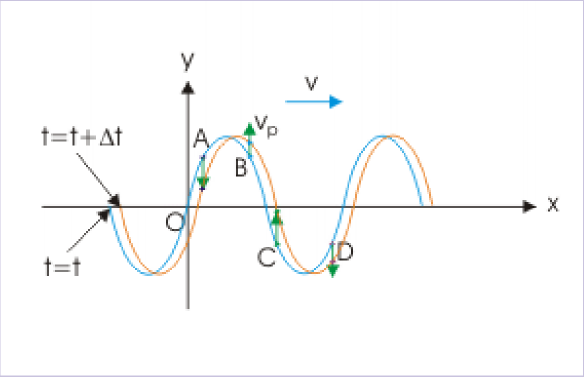


Particle velocity and slope of waveform.

The slope at “A” is positive and hence particle velocity is negative. It means that particle at this position - at the instant waveform is captured in the figure - is moving towards mean (or equilibrium) position. The slope at “B” is negative and hence particle velocity is positive. It means that particle at this position - at the instant waveform is captured in the figure - is moving towards positive extreme position. The slope at “C” is negative and hence particle velocity is positive. It means that particle at this position - at the instant waveform is captured in the figure - is moving towards mean (or equilibrium) position. The slope at “D” is positive and hence particle velocity is negative. It means that particle at this position - at the instant waveform is captured in the figure - is moving towards negative extreme position.

We can crosscheck or collaborate the deductions drawn as above by drawing wave form at another close instant t = t+Δt. We can visualize the direction of velocity by assessing the direction in which the particle at a position has moved in the small time interval considered.

Particle velocity



Particle velocity and slope of waveform.

Different forms of wave function

Different forms give rise to a bit of confusion about the form of wave function. The forms used for describing waves are :

$$y(x,t) = A\sin(kx - \omega t)$$

$$y(x,t) = A\sin(\omega t - kx)$$

Which of the two forms is correct? In fact, both are correct so long we are in a position to accurately interpret the equation. Starting with the first equation and using trigonometric identity :

$$\sin\theta = \sin(\pi - \theta)$$

We have,

$$\Rightarrow A\sin(kx - \omega t) = A\sin(\pi - kx + \omega t) == A\sin(\omega t - kx + \pi)$$

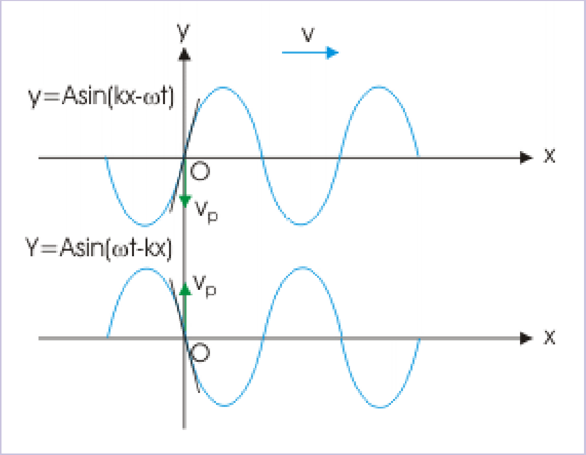
Thus we see that two forms represent waves moving at the same speed ($v = \omega/k$). They differ, however, in phase. There is phase difference of "π". This has implication on the waveform and the manner particle oscillates at any given time instant and position. Let us consider two waveforms at x=0, t=0. The slopes of the waveforms are :

$$\frac{\partial}{\partial x} y(x,t) = kA\cos(kx - \omega t) = kA = \text{a positive number}$$

and

$$\frac{\partial}{\partial x} y(x,t) = -kA\cos(\omega t - kx) = -kA = \text{a negative number}$$

Forms of wave functions



Exchange of terms in the argument of sine function results in a phase difference of π.

In the first case, the slope is positive and hence particle velocity is negative. It means particle is moving from reference origin or mean position to negative extreme position. In the second case, the slope is negative and hence particle velocity is positive. It means particle is moving from positive extreme position to reference origin or mean position. Thus two forms represent waves which differ in direction in which particle is moving at a given position.

Once we select the appropriate wave form, we can write wave equation in other forms as given here :

$$y(x,t) = A\sin(kx - \omega t) = A\sin k\left(x - \frac{\omega}{kt}\right) = A\sin\frac{2\pi}{\lambda}(x - vt)$$

Further, substituting for “k” and “ω” in wave equation, we have :

$$y(x,t) = A\sin\left(\frac{2\pi}{\lambda}x - \frac{2\pi}{T}t\right) = A\sin 2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right)$$

If we want to represent waveform moving in negative “x” direction, then we need to replace “t” by “-t”.

Example:

Problem : A wave of 50 Hz is moving along a taut string is represented by equation:

$$y(x, t = 0.001) = 0.002\cos(25\pi x)$$

The quantities are in SI units in the equation. If the wave is moving in positive x-direction, then find the speed of the wave and initial phase.

Solution : We observe that wave equation contains only “x” term at t = 0.001s. But time term with t=0.001 can not be zero unless there is initial phase. It means that we need to compare the given wave equation with the form containing initial phase :

$$y(x,t) = A\sin(kx - \omega t + \varphi)$$

According to question,

$$-\omega X 0.001+\varphi=0$$

$$\Rightarrow \varphi=0.001\omega$$

Here, we see that amplitude and wave numbers are :

$$A=0.002 \quad m \quad \text{and} \quad k=25\pi \quad \text{rad}/m$$

Clearly, we can find the speed if we know the angular frequency. Here,

$$\Rightarrow \omega=2\pi\nu=2\pi X 50=100\pi \quad \text{rad}/s$$

The speed of the wave is :

$$\Rightarrow v=\omega/k=\frac{100\pi}{25\pi}=4 \quad m/s$$

The initial phase is given as :

$$\Rightarrow \varphi=0.001\omega=0.001 X 100\pi=0.1\pi$$

Note: It may be emphasized here that the properties of string has not yet entered into our consideration. As such, the description up to this point is general to any transverse harmonic motion.

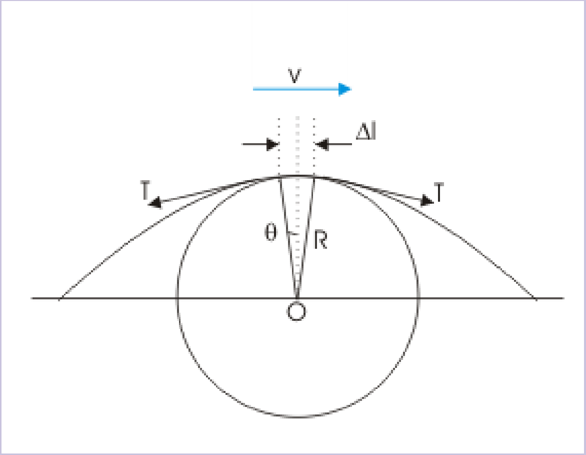
Speed of wave along a taut string

The wave speed, in general, depends on the medium of propagation and its state. In the case of string wave, the wave motion along a stretched string approximates harmonic transverse wave under simplified assumptions. We consider no loss of energy as the wave moves along the string.

For a simplified consideration, we assume that inertial property (mass per unit length) is small and uniform. A light string is important as we consider uniform tension in the string for our derivation. It is possible only if the string is light. When the string changes its form, the elastic force tends to restore deformation without any loss of energy. This means that we are considering a perfectly elastic deformation.

The mechanical transverse wave speed on a taut string is a function of the inertial property (mass per unit length) and tension in the string. Here, we shall investigate the motion to drive an expression of speed. For this we consider a single pulse and a small element of length Δl as shown in the figure. Let mass per unit length be μ, then the mass of the element is :

Wave speed



An element of string and forces on it.

$$\Delta m = \mu \Delta l$$

The small element is subjected to tensions in the string. The element does not move sideways along the circular path in the case of transverse wave motion. Rather it is the wave form which approximates to a curved path. For our mechanical analysis, however, the motion of wave is equivalent to motion of small string along the curved path having a radius of curvature “R” moving at a speed equal to the wave speed, “v”. Thus, we can analyze wave motion by assuming as if the small element is moving along a circular path for a brief period. Now, the string wave moves with constant speed and as such there is no tangential acceleration associated with the analysis. There is only centripetal acceleration in the radial direction.

The centripetal force is equal to the resultant of tensions on either side of the element and is given by :

$$F = 2T \sin\theta$$

For small angle involved, we can approximate as :

$$\sin\theta = \frac{\frac{\Delta l}{2}}{R} = \frac{\Delta l}{2R}$$

Combining above two equations, we have :

$$\Rightarrow F = 2T \sin\theta = \frac{2T\Delta l}{2R} = \frac{T\Delta l}{R}$$

On the other hand, the centripetal force for uniform circular motion is given by :

$$F = \Delta m a_R = \frac{\mu \Delta l v^2}{R}$$

Equating two expressions of centripetal force, we have :

$$\Rightarrow \frac{\mu \Delta l v^2}{R} = \frac{T\Delta l}{R}$$

$$\Rightarrow v = \sqrt{\left(\frac{T}{\mu}\right)}$$

It is evident from the expression that speed on the stretched string depends on inertial factor (μ) and tension in the string (T)

Dependencies

We need to create a simple harmonic vibration at the free end of the string to generate a harmonic transverse wave in the string. The frequency of wave, therefore, is determined by the initial condition that sets up the vibration. This is a very general deduction applicable to waves of all types. For example, when light wave (electromagnetic non-mechanical wave) passes through a glass of certain optical property, then speed of the light changes along with change in wavelength – not change in frequency. The frequency is set up by the orbiting electrons in the atoms, which radiates light. Thus, we see that frequency and time period (inverse of frequency) are medium independent attributes.

On the other hand, speed of the wave is dependent on medium or the material along which it travels. In the case of string wave, it is dependent on mass per unit length (inertial attribute) and tension in the string. The speed, in turn, determines wavelength in conjunction with time period such that :

$$\Rightarrow \lambda = vT$$

One interesting aspect of the dependencies is that we can change wave speed and hence wavelength by changing the tension in the string. We should note here that we have considered uniform tension, which may not be realized in practice as strings have considerable mass. Cleary such situation will lead to change in speed and wavelength of the wave as it moves along the string. On the other hand, the amplitude of wave is a complex function of initial condition and the property of wave medium.

Example:

Problem : A string wave of 20 Hz has an amplitude of 0.0002 m along a string of diameter 2 mm. The tension in the string is 100 N and mass density of the string is $2X10^3 \quad kg/m^3$. Write equation describing wave moving in x-direction.

Solution : : The equation of a wave in x-direction is given as :

$$y(x,t) = A\sin(kx - \omega t)$$

The amplitude of wave is given. Hence,

$$\Rightarrow y(x,t) = 0.0002\sin(kx - \omega t)$$

Now, we need to know “k” and ”ω” to complete the equation. The frequency of the wave is given. The angular frequency, therefore, is :

$$\Rightarrow \omega = 2\pi\nu = 2X3.14X20 = 125.6 \quad \text{radian}/s$$

In order to determine wave number “k”, we need to calculate speed. An inspection of the unit of mass density reveals that it is volume density of the string – not linear mass density as required. However, we can convert the same to linear mass density as string is an uniform cylinder.

$$\text{linear mass density} = \text{volume mass density} X \text{cross section area}$$

$$\Rightarrow \mu = 2X10^3 X\frac{\pi d^2}{4} = 2X10^3 X\frac{3.14X(0.002)^2}{4}$$

$$\Rightarrow \mu = 6.28X10^{-3} \quad kg/m$$

Putting in the expression of speed, we have :

$$\Rightarrow v = \sqrt{(T/\mu)} = \sqrt{\left(\frac{100}{6.28X10^{-3}}\right)} = \sqrt{\left(\frac{10^5}{6.28}\right)} = 39 \quad m/s$$

The wave number, k, is :

$$\Rightarrow k = \frac{\omega}{v} = \frac{125.6}{39} = 3.22 \quad \text{rad}\,/m$$

Substituting in the wave equation,

$$y(x,t) = 0.0002\sin(3.22x - 125.6t)$$

Energy transmission by waves

Wave represents distributed energy. Except for standing waves, the wave transports energy along with it from one point to another. In the context of our course, we shall focus our attention to the transverse harmonic wave along a string and investigate energy being transported by it. But the discussion and results would be applicable to other transverse waves as well through a medium. For a base case, we shall consider an ideal case in which there is no loss of energy.

We supply energy by continuously vibrating the free end of taut string. This energy is transmitted by the small vibrating string element to the neighboring element following it. We can see that vibrating small elements (also referred as particle) possess energy as it oscillates (simple harmonic motion for our consideration) in the transverse direction. It can be easily visualized as in the case of SHM that the energy has two components i.e. kinetic energy (arising from motion) and potential energy (arising from position). The potential energy is elastic potential energy like that of spring. The small string element is subjected to tension and thereby periodic elongation and contraction during the cycle of oscillatory motion.

There are few important highlights of energy transport. The most important ones are :

- The energy at the extreme positions along the string corresponds to zero energy.
- The kinetic and elastic potential energies at the mean (equilibrium) position are maximum.

Kinetic energy

The velocity of string element in transverse direction is greatest at mean position and zero at the extreme positions of waveform. We can find expression of transverse velocity by differentiating displacement with respect to time. Now, the y-displacement is given by :

$$y = A\sin(kx - \omega t)$$

Differentiating partially with respect to time, the expression of particle velocity is :

$$v_p = \frac{\partial y}{\partial t} = -\omega A \cos(kx - \omega t)$$

In order to calculate kinetic energy, we consider a small string element of length “dx” having mass per unit length “μ”. The kinetic energy of the element is given by :

$$dK = \frac{1}{2} dm v_p^2 = \frac{1}{2} \mu dx \omega^2 A^2 \cos^2(kx - \omega t)$$

This is the kinetic energy associated with the element in motion. Since it involves squared cosine function, its value is greatest for a phase of zero (mean position) and zero for a phase of π/2 (maximum displacement). Now, we get kinetic energy per unit length, “ K_L ”, by dividing this expression with the length of small string considered :

$$K_L = \frac{dK}{dx} = \frac{1}{2} \mu \omega^2 A^2 \cos^2(kx - \omega t)$$

Rate of transmission of kinetic energy

The rate, at which kinetic energy is transmitted, is obtained by dividing expression of kinetic energy by small time element, “dt” :

$$\frac{dK}{dt} = \frac{1}{2} \mu \frac{dx}{dt} \omega^2 A^2 \cos^2(kx - \omega t)$$

But, wave or phase speed,v, is time rate of position i.e. $\frac{dx}{dt}$. Hence,

$$\frac{dK}{dt} = \frac{1}{2} \mu v \omega^2 A^2 \cos^2(kx - \omega t)$$

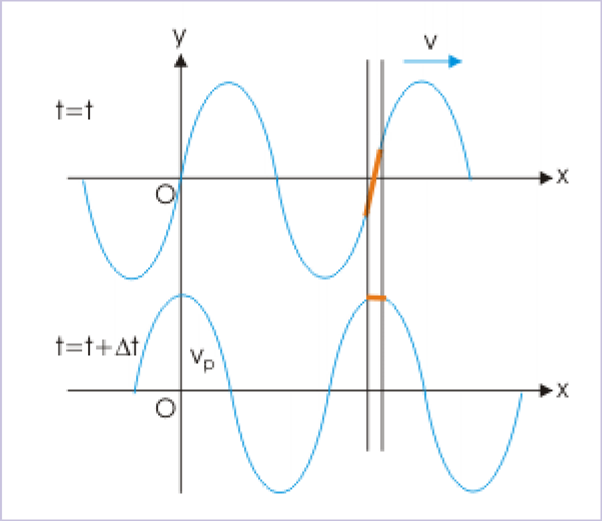
Here kinetic energy is a periodic function. We can obtain average rate of transmission of kinetic energy by integrating the expression for integral wavelengths. Since only $\cos^2(kx - \omega t)$ is the varying entity, we need to find average of this quantity only. Its integration over integral wavelengths give a value of “1/2”. Hence, average rate of transmission of kinetic energy is :

$$\left.\frac{dK}{dt}\right|_{\text{avg}} = \frac{1}{2} X \frac{1}{2} \mu v \omega^2 A^2 = \frac{1}{4} \mu v \omega^2 A^2$$

Elastic potential energy

The elastic potential energy of the string element results as string element is stretched during its oscillation. The extension or stretching is maximum at mean position. We can see in the figure that the length of string element of equal x-length “dx” is greater at mean position than at the extreme. As a matter of fact, the elongation depends on the slope of the curve. Greater the slope, greater is the elongation. The string has the least length when slope is zero. For illustration purpose, the curve is purposely drawn in such a manner that the elongation of string element at mean position is highlighted.

Elongation in the string



The string element stretched most at equilibrium position.

Greater extension of string element corresponds to greater elastic energy. As such, it is greatest at mean position and zero at extreme position. This deduction is contrary to the case of SHM in which potential energy is greatest at extreme positions and zero at mean position.

The elastic potential energy of an element of mass “dm” is given by the expression :

$$dU = \frac{1}{2}\mathrm{d}mv^2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2$$

where “v” is wave speed and “ $\frac{\mathrm{d}y}{\mathrm{d}x}$ ” is the slope of waveform. The respective expressions for these quantities are :

$$v = \frac{\omega}{k}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = kA\cos(kx - \omega t)$$

Putting in the expression of elastic potential energy,

$$dU = \frac{\mu \mathrm{d}x\omega^2 k^2 A^2 \cos^2(kx - \omega t)}{2k^2} = \frac{1}{2}\mu \mathrm{d}x\omega^2 A^2 \cos^2(kx - \omega t)$$

This is the same expression as derived for kinetic energy. The elastic potential energy per unit length is :

$$U_L = \frac{\mathrm{d}U}{\mathrm{d}x} = \frac{1}{2}\mu\omega^2 A^2 \cos^2(kx - \omega t)$$

Rate of transmission of elastic potential energy

The rate, at which elastic potential energy is transmitted, is obtained by dividing expression of kinetic energy by small time element, “dt”. This expression is same as that for kinetic energy.

$$\frac{dU}{dt} = \frac{1}{2}\mu v\omega^2 A^2 \cos^2(kx - \omega t)$$

and average rate of transmission of elastic potential energy is :

$$\left.\frac{dU}{dt}\right|_{\text{avg}} = \frac{1}{2} X \frac{1}{2}\mu v\omega^2 A^2 = \frac{1}{4}\mu v\omega^2 A^2$$

Mechanical energy

Since the expression of elastic potential energy is same as that of kinetic energy, we get mechanical energy expressions by multiplying expression of kinetic energy by “2”. The mechanical energy associated with small string element, "dx", is :

$$dE = 2XdK = 2X\frac{1}{2}dmv_p^2 = \mu dx\omega^2 A^2 \cos^2(kx - \omega t)$$

Similarly, the mechanical energy per unit length is :

$$E_L = 2X\frac{1}{2}\mu\omega^2 A^2 \cos^2(kx - \omega t) = \mu\omega^2 A^2 \cos^2(kx - \omega t)$$

Rate of transmission of mechanical energy

Mechanical energy transmitted by the wave is sum of kinetic energy and elastic potential energy. The instantaneous rate of transmission of mechanical energy, therefore, is also obtained by multiplying instantaneous kinetic energy by “2”.:

$$\frac{dE}{dt} = 2X\frac{dK}{dt} = \mu v\omega^2 A^2 \cos^2(kx - \omega t)$$

Energy is greatest at the mean position and zero at the extreme positions of the particle in SHM. How can this happen? We need to interpret this fact in the light of transmission of energy from one section of the string to another. When element position changes in transverse direction away from mean position, it passes on the energy to the neighboring element. In doing so, the energy of string element is reduced, whereas the particle following it in the direction of wave motion gains energy (we consider no energy loss in the process). In more direct words, we can say that elements pass on the disturbance to the adjacent element. As the element moves to the extreme position, it completely exhausts its energy and acquires its undisturbed length. However, it yields to next train of energy or disturbance as the free end of the string is moved again and again.

Average power transmitted

The average power transmitted by wave is equal to time rate of transmission of mechanical energy over integral wavelengths. It is equal to :

$$P_{\text{avg}} = \left.\frac{dE}{dt}\right|_{\text{avg}} = 2X\frac{1}{4}\mu v\omega^2 A^2 = \frac{1}{2}\mu v\omega^2 A^2$$

If mass of the string is given in terms of mass per unit volume, “ρ”, then we make appropriate change in the derivation. We exchange “μ” by “ρs” where “s” is the cross section of the string :

$$P_{\text{avg}} = \frac{1}{2}\rho s v\omega^2 A^2$$

Energy density

Since there is no loss of energy involved, it is expected that energy per unit length is uniform throughout the string. As much energy enters that much energy goes out for a given length of string. This average value along unit length of the string length is equal to the average rate at which energy is being transferred.

The average mechanical energy per unit length is equal to integration of expression over integral wavelength :

$$E_L\Big|_{\text{avg}} = 2X\frac{1}{4}\mu v\omega^2 A^2 = \frac{1}{2}\mu v\omega^2 A^2$$

We have derived this expression for harmonic wave along a string. The concept, however, can be extended to two or three dimensional transverse waves. In the case of three dimensional transverse waves, we consider small volumetric element. We, then, use density,ρ, in place of mass per unit length, μ. The corresponding average energy per unit volume is referred as energy density (u):

$$u = \frac{1}{2}\rho v\omega^2 A^2$$

Intensity

Intensity of wave (I) is defined as power transmitted per unit cross section area of the medium :

$$I = \rho s v\omega^2 \frac{A^2}{2} s = \frac{1}{2}\rho v\omega^2 A^2$$

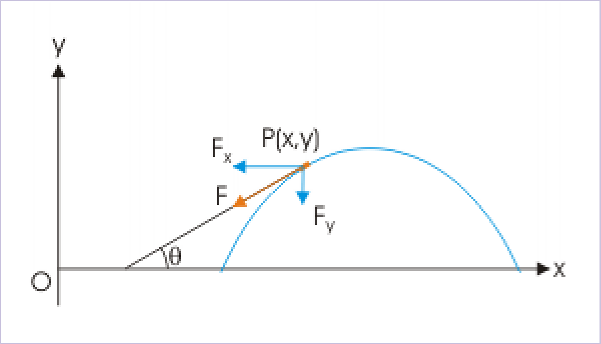
Intensity of wave (I) is a very useful concept for three dimensional waves radiating in all direction from the source. This quantity is usually referred in the context of light waves, which is transverse harmonic wave in three dimensions. Intensity is defined as the power transmitted per unit cross sectional area. Since light spreads uniformly all around, intensity is equal to power transmitted, divided by spherical surface drawn at that point with source at its center.

Mechanics of energy transmission

Derivation of expression of energy transmission considering classical concept of power gives an insight into the manner energy is transmitted along the string. We have pointed out that every element pulls element ahead. In this fashion, a string element is worked out by the adjacent element. In this fashion, energy is transmitted from one element preceding to the element following it.

We consider a small string element as shown in the figure. Here, our objective is to evaluate the basic power equation as given :

Work by tension



The string element is pulled by tension in the string.

$$P = \text{Force} X \text{velocity}$$

Now, the string element is pulled by tension “F” as shown in the figure. The y-component of force in the direction of oscillation is :

$$F_y = -F \sin \theta$$

Since angle “θ” is small, sinθ = tanθ. But tanθ is slope of the waveform at x=x. Therefore,

$$Fy = -F \frac{dy}{dx}$$

$$Fy = -F\frac{\mathbb{d}}{\mathbb{d}x}A\sin(kx-\omega t) = -FkA\cos(kx-\omega t)$$

On the other hand, velocity of the small element is :

$$v_p = \frac{\partial y}{\partial t} = -\omega A\cos(kx-\omega t)$$

Putting these expressions, the power is given as :

$$P = \text{Force}X\text{velocity}$$

$$P = -F\left(\frac{\mathbb{d}y}{\mathbb{d}x}\right)X\left(\frac{\mathbb{d}y}{\mathbb{d}t}\right)$$

$$P = -FkA\cos(kx-\omega t)X-\omega A\cos(kx-\omega t)$$

$$P = F\omega kA^2\cos^2(kx-\omega t)$$

Using $v = \sqrt{\left(\frac{F}{\mu}\right)}$ and $v = \frac{\omega}{k}$,

$$P = F\omega kA^2\cos^2(kx-\omega t) = \mu v^2\omega\left(\frac{\omega}{v}\right)A^2\cos^2(kx-\omega t)$$

$$P = \mu v\omega^2A^2\cos{}^2(kx-\omega t)$$

The average transmission of power is obtained by integrating the expression over integral wavelength,

$$P_{\text{avg}} = \frac{1}{2}\mu v\omega^2A^2$$

This expression is same as the expression obtained earlier by adding kinetic and elastic potential energy.